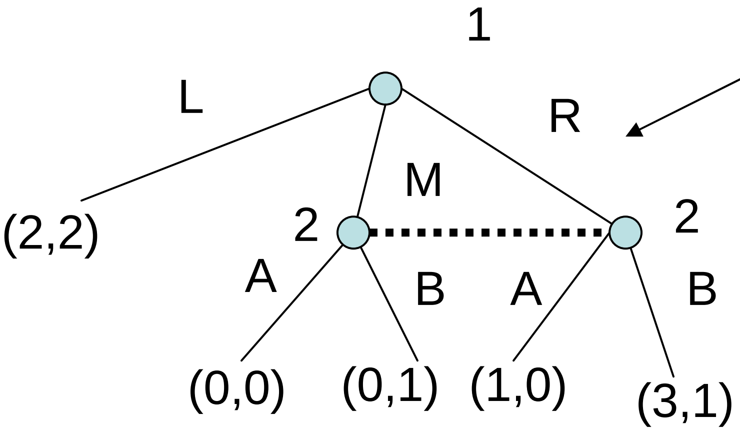
Dynamic games of incomplete information

- Recall: static games of incomplete information
  - The game/payoffs depend on the type of players. A player knows its own type but it does not know the types of the other players.
  - Transform a game of incomplete information → game of imperfect information
    - Assign probabilities for the types of the players
    - Perceived as a move by nature
    - Represents the players' *apriori* belief on the types of other players
- What changes for the dynamic game?
  - Players have the chance of updating their beliefs based on the observed actions of the other players.

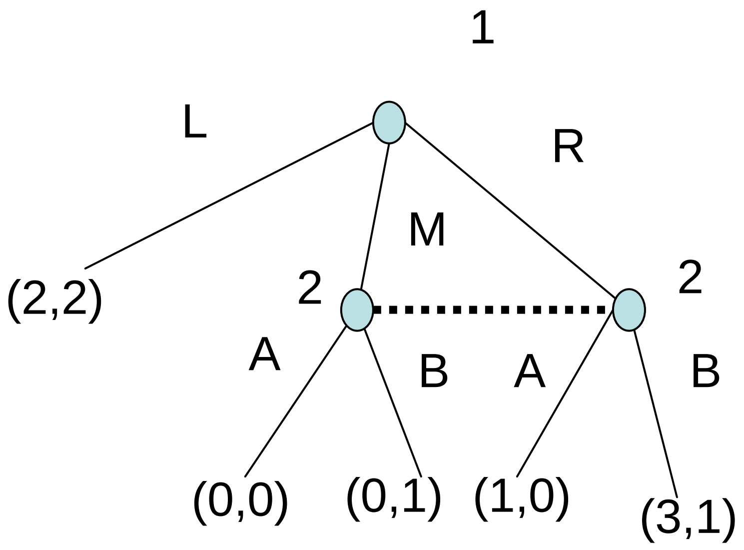
Subgame perfection for dynamic games of incomplete information?

- The concept of subgame perfection → harder to apply for games of incomplete information
  - Start of a period does not form a well-defined subgame
  - Formally: the only proper subgame of a game of incomplete information is the whole game → any Nash equilibrium is subgame perfect
  - To illustrate that: consider the following example of a game of imperfect information



What are the Nash eq. for this game?

Subgame perfection



- Nash equilibria: (L,A) and (R,B)
- Both subgame perfect

However, (L,A) is a sub-optimal action → may be ruled for the case of sequential equilibrium

Perfect Bayesian eq. for multi-stage games

- **The Basic Signaling Game**
  - Simplest type of game: 2 players
    - Player 1: leader (sender)
    - Player 2: follower (receiver)
    - Player 1 has private inf. about its type $\theta \in \Theta$, action $a_1 \in A_1$
    - Player 2: its type is common knowledge, action $a_2 \in A_2$
    - Space of mixed actions: $\mathcal{A}_1$ and $\mathcal{A}_2$, with elements α_1 and α_2
    - Utility of player i: $u_i(\alpha_1, \alpha_2, \theta)$
    - Player 2 – prior belief about player's 1 type: *p* → common knowledge
    - Strategy for player 1: probability distribution $\sigma_1(.| \theta)$ over actions $a_1 \in A_1$ for each type θ
    - Strategy for player 2: probability distribution $\sigma_1(.| \theta)$ over actions $a_2 \in A_2$ for each $a_1 \in A_1$

Basic signaling game: payoffs

- Payoff for player 1, given its type θ:

$$u_1(\sigma_1,\sigma_2,\theta)=\sum_{a_1}\sum_{a_2}\sigma_1(a_1\mid\theta)\sigma_2(a_2\mid a_1)u_1(a_1,a_2,\theta)$$

- Player 2's (ex ante – beforehand payoff) to strategy $\sigma_2(.|a_1)$, when player 1 plays $\sigma_1(.|\ \theta)$:

$$\sum_{\theta}p(\theta)\left[\sum_{a_1}\sum_{a_2}\sigma_1(a_1\mid\theta)\sigma_2(a_2\mid a_1)u_2(a_1,a_2,\theta)\right]$$

- Should we make the decision based on the above computed payoff?

Posterior beliefs

- Player 2, observes the action of player 2 → must update its belief on θ, and its choice of action → posterior distribution over Θ: $\mu(.|a_1)$
- How to compute $\mu(.|a_1)$?
  - Player 1 actions may depend on its type
  - Let $\sigma_1{}^*(.|\ \theta)$ to be player 1 strategy
  - Know p(.), $\sigma_1{}^*(.|\ \theta)$ and observe a_1: use Bayes rule
- Extension of subgame-perfect eq. → perfect Bayesian eq.
  - Player 2 max. its payoff conditional on a_1:

$$\sum_{\theta} \mu(\theta \mid a_1) u_2\left(a_1, \sigma_2(. \mid a_1), \theta\right) = \sum_{\theta} \sum_{a_2} \mu(\theta \mid a_1) \sigma_2(a_2 \mid a_1) u_2\left(a_1, a_2, \theta\right)$$

Perfect Bayesian Equilibrium (PBE)

- **Definition:** A perfect Bayesian eq. of a signaling game is a strategy profile σ* and posterior beliefs $\mu(.|a_1)$, s.t.:

- (P1): $\forall\theta,\ \sigma_1^*(.|\theta) \in \arg\max_{\alpha_1} u_1(\alpha_1, \sigma_2^*, \theta)$

- (P2): $\forall a_1,\ \sigma_2^*(.|a_1) \in \arg\max_{\alpha_2} \sum_{\theta} \mu(\theta|a_1) u_2(a_1, \alpha_2, \theta)$

- (B) $$\mu(\theta|a_1) = \frac{p(\theta)\sigma_1^*(a_1|\theta)}{\sum_{\theta'\in\Theta} p(\theta')\sigma_1^*(a_1|\theta')}, \quad \text{if} \quad \sum_{\theta'\in\Theta} p(\theta')\sigma_1^*(a_1|\theta') > 0$$

μ(.|a1), is any prob. distr. on Θ, if $\sum_{\theta'\in\Theta} p(\theta')\sigma_1^*(a_1|\theta') = 0$

Backward induction solution?

- Backward induction was used in games with perfect information
- PBE: strategies are optimal given the beliefs and the beliefs are obtained from equilibrium strategies and observed actions
  - Circularity → PBE cannot be determined by backward induction

An example of a signaling game

- **Two period reputation game**
  - 2 firms on the market

  Period 1: both firms of the market; only firm 1 action a1
  - Actions for firm 1: prey or accommodate
    - If prey: firm 2 gets P2<0
    - If accommodate: firm 2 gets D2>0
  - Type of firm 1: “sane” or “crazy”
    - “crazy” – always prey
    - “sane”
      - If accommodates: payoff for 1: D1>0
      - If preys: payoff for 1: P1< D1 → prefers to accommodate
      - If 2 exits→ M1>D1 (monopoly)

Two period reputation game: cont.

- Period 2 - player 2 selects action a2: stay or exit
  - If exits, it gets a 0 payoff, and player 1 gets M1> D1
- Assumptions:
  - Player 1 knows his type
  - Player 2 believes that player 1 is sane with probability p
  - δ = discount factor between the two periods
- Building reputation for the sane player
  - Player 1 may try to convince player 2 that he is crazy, to get M1>D1 in the second period of the game.

Taxonomy of PBE

- **Separating equilibrium:** the two types of player 1, choose two different actions in period 1
  - Firm 2 has complete information for the second period

$$\mu(\theta = \text{sane} \mid a_1 = \text{accomodate}) = 1$$

$$\mu(\theta = \text{crazy} \mid a_1 = \text{prey}) = 1$$

- **Pooling equilibrium:** the two types of player 1, choose the same action in period 1

$$\mu(\theta = \text{sane} \mid a_1 = \text{prey}) = p$$

- **Hybrid (semi-separating equilibria):** the sane type may randomize between preying and accommodating

$$\mu(\theta = \text{sane} \mid a_1 = \text{prey}) \in (0, p)$$

$$\mu(\theta = \text{sane} \mid a_1 = \text{accommodate}) = 1$$

What type of equilibrium? Existence.

- Separating eq. existence: sufficient and necessary condition

$$D_1(1+\delta) \geq P_1 + \delta M_1 \Leftrightarrow \delta(M_1 - D_1) \leq (D_1 - P_1)$$

- Pooling eq. → to enforce exit for player 2. Condition:

$$pD_2 + (1-p)P_2 \leq 0$$

- If the above two conditions do not hold → hybrid PBE

Note: uniqueness of the eq. in this case, is due to the fact that the “crazy” gtype is assumed to always prey.

Multi-stage games with observed actions and incomplete information

- Each player i has type θ_i, and types are independent

$$p(\theta) = \prod_{i=1}^{I} p_i(\theta_i)$$

- At each period t (t=0,1,2,… T), players choose their actions simultaneously, and the actions are revealed at the end of the period
- Players' action set at a date t is type independent
- Behavior strategy: $\sigma_i(a_i \mid h^t, \theta)$
- Payoffs $u_i(h^{T+1}, \theta)$

- Subgame perfection → BNE not only for the whole game, but also for the "continuation game" starting at period t after all possible histories h^t
  - Continuation games → proper subgames?
    - No. They do not stem from a singleton information set

Continuation games → true games

- Need to specify the players' beliefs at the start of each continuation game.
- **Definition:** A perfect Bayesian equilibrium is a (σ,μ) that satisfies (P) and (B(i) – B(iv)).

B(i) Posterior beliefs are independent, and all types of player i have the same belief.

- For all θ, t and h^t:

$$\mu_i(\theta_{-i} \mid \theta_i, h^t) = \prod_{j \neq i} \mu_i(\theta_j \mid h^t)$$

- even unexpected events will not change the independence assumption for the type of the opponents

Perfect Bayesian equilibrium: cont

- B(ii) Beliefs are updated according to Bayes' rule:
  - For all i,j, h^t, and a_j^t, If there exist

$$\hat{\theta}_j,\ \text{s.t. } \mu_i\left(\theta_j \mid h^t\right) > 0,\ \sigma_{\mathrm{j}}\left(a_j^t \mid h^t, \theta_j\right) > 0,\ \text{then}$$

$$\mu_i\left(\theta_j \mid h^t, a^t\right) = \frac{\mu_i\left(\theta_j \mid h^t\right)\sigma_j\left(a_j^t \mid h^t, \theta_j\right)}{\sum_{\hat{\theta}_j} \mu_i\left(\hat{\theta}_j \mid h^t\right)\sigma_j\left(a_j^t \mid h^t, \hat{\theta}_j\right)}$$

- B(iii) Don't signal what you don't know
  - For all i,j, h^t, and a^t and $\hat{a}^t$

$$\mu_i\left(\theta_j \mid (h^t, a^t)\right) = \mu_i\left(\theta_j \mid (h^t, \hat{a}^t)\right),\ \text{if } a_j^t = \hat{a}_j^t$$

Perfect Bayesian eq. – cont.

- B(iv) All players have to have the same belief about the type of another player
  - Imposed because of the req. of eq. analysis: players have the same belief about each other’s strategies.
  - For all θ_k, and h^t

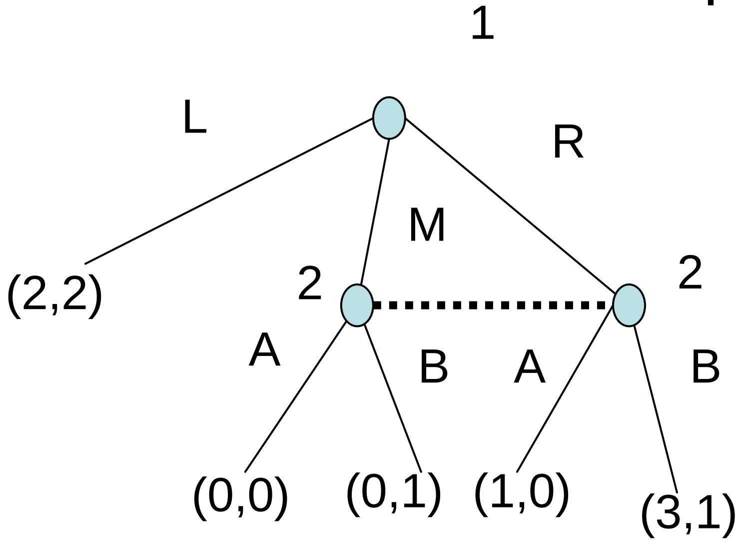
$$\mu_i\left(\theta_k \mid h^t\right) = \mu_k\left(\theta_k \mid h^t\right) = \mu\left(\theta_k \mid h^t\right), \text{ for } i \neq j \neq k$$

- (P) For each player i, type θ_i, alternative strategy $\sigma'_{I,}$, and history h^t

$$u_i\left(\sigma \mid h^t, \theta_i, \mu(. \mid h^t)\right) \geq u_i\left(\sigma'_i, \sigma_{-i} \mid h^t, \theta_i, \mu(. \mid h^t)\right)$$

Sequential equilibrium

- We saw already that the requirement that the players' strategies form a Nash equilibrium is too weak → formally the only proper subgame for the games of incomplete, or imperfect information is the whole game.
- Recall the initial example



- Nash equilibria: (L,A) and (R,B)
- Both subgame perfect

Is (LA) equilibrium plausible?

Sequential equilibrium: cont.

- (L,A) is not plausible
  - Whatever player's 2 beliefs on player's 1 move (M or R), he must chose B if he has an opportunity to move.

- Need to generalize the previous condition (P) → given the system of beliefs, no player can gain by deviating at any information set.
- (s) An assessment (σ,μ) is *sequentially rational* if, for any information set h, and alternative strategy $\sigma'_{i(h)}$,

$$u_{i(h)}\left(\sigma \mid h, \mu(h)\right) \geq u_{i(h)}\left(\sigma'_{i(h)}, \sigma_{-i(h)} \mid h, \mu(h)\right)$$

Sequential eq. cont.

- Consistency condition on beliefs is also introduced

- (C) An assessment (σ,μ) is consistent if

$$(\sigma, \mu) = \lim_{n \to +\infty} (\sigma^n, \mu^n)$$

For some sequence $(\sigma^n, \mu^n) \in \Psi^0$ ⟵ The set of all assessments

- **Definition:** A sequential equilibrium is an assessment (σ,μ) that satisfies (S) and (C)

Existence: For any finite extensive-form game, there exist at least one sequential equilibrium.

Learning in games

- Why learning?
  - For introspection, the rules of the game, rationality of the players, payoff functions – all common knowledge
  - Another problem: for multiple equilibria, how players come to expect the same equilibrium?

- Applicability
- Repeated games
- Teach opponent to play a best response to a particular action, by repeating it over and over again

Example of sophisticated learning

- How would you play this game, if you were player 1?

	L	R
U	1,0	3,2
D	2,1	4,0

Sophisticated learning?

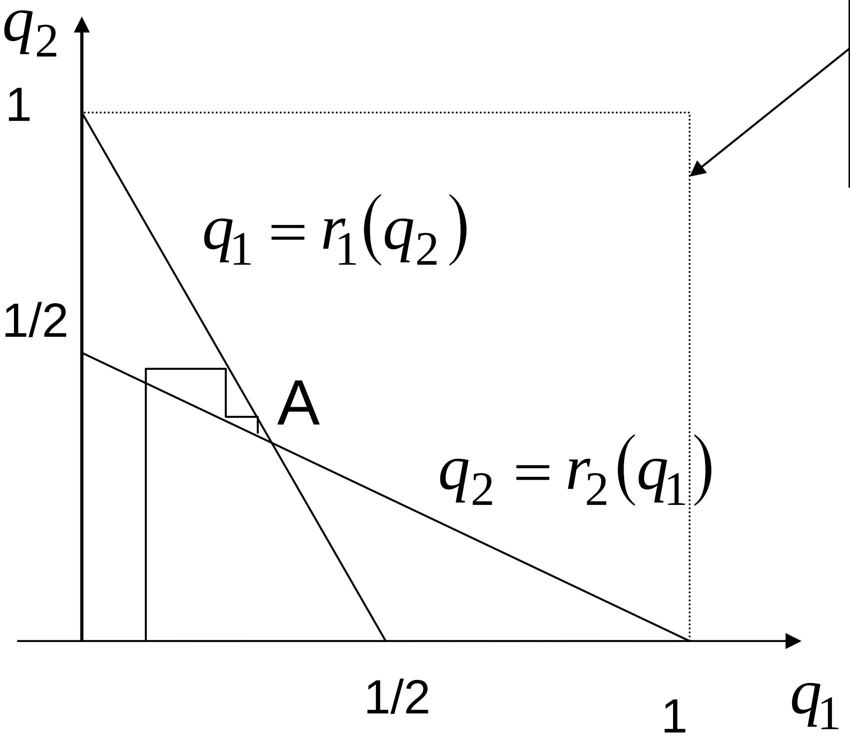
- Most learning theory → models for which the incentive is small to alter the future play of the opponents.
  - Examples:
    - large anonymous population: population size large compared to the discount factor
    - Players locked in their choices and discount factor small compared to maximum speed at which the system can possibly adjust

Common models for learning

- **Fictitious play**
  - Players observe only their own matches and play a best response to the historical frequency of play
- **Partial best-response**
  - A fixed portion of users switches each period from its current action to a BR to the aggregate statistics from the previous period
- **Replicator Dynamics**
  - The fraction of the population using a given strategy, grows proportionally to that strategy’s current payoff.

One type of learning: Cournot adjustment

- Unique Nash eq. is at the intersection of the reaction curves



The process converges to
Nash equilibrium from any
Starting point →eq. globally stable

Fictitious play

- Repeated game
- Stationary assumption
- Each player: belief of opponents “strategy” by looking at what happened
- Player then plays best response (BR) according the his belief
- Belief: a prediction of the opponent action distribution, i.e. the degree to which player i believes player j will play a certain action.
- Players choose their actions in each period , s.t. to maximize their expected payoff, with respect to their belief for the current period.

Updating beliefs

- Player i: initial weight function

$$K_0^i : S^{-i} \rightarrow \mathcal{R}^+$$

- Game iteratively repeated → K updated:

$$K_t(s^{-i}) = K_{t-1}(s^{-i}) + \begin{cases} 1, & \text{if } s_{t-1}^{-i} = s^{-i} \\ 0, & \text{ow.} \end{cases}$$

- Given the frequency vector K → updates beliefs
  - The belief player i has at time t about its opponent to play s^{-i} at time t:

$$\gamma_t^i\left(s^{-i}\right) = \frac{K_t^i\left(s^{-i}\right)}{\sum_{\hat{s} \in S^{-i}} K_t^i\left(\hat{s}^{-i}\right)} \longleftarrow$$

Simple normalization

Fictitious play – recall from last lecture

- Player i: initial weight function

$$K_0^i : S^{-i} \rightarrow \mathcal{R}^+$$

- Game iteratively repeated → K updated:

$$K_t(s^{-i}) = K_{t-1}(s^{-i}) + \begin{cases} 1, & \text{if } \; s_{t-1}^{-i} = s^{-i} \\ 0, & \text{ow.} \end{cases}$$

- Given the frequency vector K → updates beliefs
  - The belief player i has at time t about its opponent to play s^{-i} at time t:

$$\gamma_t^i\left(s^{-i}\right) = \frac{K_t^i\left(s^{-i}\right)}{\sum\limits_{\hat{s} \in S^{-i}} K_t^i\left(\hat{s}^{-i}\right)} \longleftarrow$$

Simple normalization

Fictitious play

- Given the updated belief γ_t^i
- Fictitious play: any rule $p_t^i(\gamma_t^i) \in BR^i(\gamma_t^i)$
- Not a unique fictitious play rule → there may be more than one best response to a particular assessment

Convergence properties for fictitious play

- Proposition:
  - (1) If s is a strict Nash equilibrium, and s is played at date t in the process of fictitious play, then s is played at all subsequent dates (Nash equilibria are absorbing for the process of fictitious play)
  - (2) Any pure-strategy steady state of fictitious play must be Nash equilibrium
- Definition: marginal empirical distribution of play:

$$d_t^j(s^j) = \frac{k_t(s^j) - k_0(s^j)}{t}$$

- Proposition:
  - If the empirical distributions over each player's choices converge, the strategy profile corresponding to the product of these distributions is a Nash eq.
- Proposition: the fictitious play process converges for a two person zero-sum game

No regret learning

- **No regret learning strategies:** probabilistic learning alg. which specify that players ***explore*** the space of actions by playing all actions with some non-zero probability, and ***exploit*** successful actions, by increasing the probability of employing those actions that generate high profits.
  - Learn mixed strategy equilibria
- No external regret algorithms
  - **External regret:** difference between the payoffs achieved by the strategies prescribed by the given algorithm, and the payoffs achieved by any other fixed sequence of decisions, in the worst case.
- No internal regret algorithms
  - **Internal regret:** difference between the payoffs achieved by the strategies prescribed by the given algorithm, and the payoffs that could be achieved by a remapped sequence of strategies.
    - Sequence is remapped, if there is a mapping f of the strategy space into itself, s.t. for each occurrence of strategy x (e.g. x=169), the mapped strategy y (e.g. y = 170) appears in the re-mapped sequence
  - No internal regret → no external regret

No External Regret Algorithm – an example alg.

- Freud & Schapire:
  - "Game theory, on-line prediction and boosting", Proc. of the 9th Annual Conference on Computational Learning Theory, pp. 325-332, ACM press, May 1996.
- No external regret via multiplicative updating
- ρ_i^t = cumulative payoff obtained through time t via strategy t+1:

$$\rho_i^t = \sum_{x=0}^{t} r_i^x$$

- The weight associated with strategy t+1, for $\beta \in [0,1)$:

$$w_i^{t+1} = \frac{(1+\beta)^{\rho_i^t}}{\sum_{j=1}^{S} (1+\beta)^{\rho_j^t}}$$

No external regret –cont.

- The weight calculation represents a measure of regret
- Need to know payoff that would be obtained for all possible strategies
- Naive player: knows only his payoff for the currently played strategy
  - Previous algorithm may be modified to work for naive players: [Auer, Chesa-Bianchi, Freud and Schapire, 1995]
    - Gambling in a rigged casino: the adversarial multi-armed bandit problem, proc. of the 36th Annual Symposium on Foundations of Computer Science, pp. 322-331. ACM Press, Nov. 1995.

No internal regret – an example alg.

- The regret for a player at time t: difference between the payoffs achieved using its strategy of choice, e.g. i, and the payoffs that could have been achieved had strategy j≠i been played instead:

$$R_{i\to j}^{t} = 1_{i}^{t}\left(r_{j}^{t} - r_{i}^{t}\right)$$

Indicator function: 1 if strategy i employed at t, 0 ow.

- Cumulative regret:

$$R_{i\to j}^{T} = \sum_{t=0}^{T} R_{i\to j}^{t}$$

- Internal regret:

$$IR_{i\to j}^{T} = (R_{i\to j}^{T})^{+}$$

$$X^{+} = \max(X,0)$$

No internal regret – cont.

- Cumulative regret for playing all strategies but j:

$$IR_{S\to j}^{T} = \sum_{i=1}^{S} IR_{i\to j}^{T}$$

- Updating weights
  - If strategy i is played at time t,

$$w_j^{t+1} = \frac{1}{\mu} IR_{i\to j}^{t}$$

$$w_i^{t+1} = 1 - \sum_{j\neq i} w_j^{t+1}$$

  - μ = normalizing term:

$$\mu > \left(|S| - 1\right) \max_{j\in S} IR_{i\to j}^{t}$$

No internal regret alg. – cont.

Some observations:

- Achieves no internal regret in the limit as T→ ∞
- Learning converges to the correlated Nash equilibrium
- Naive version also proposed
- Reference:
  - S. Hart & A. Mass Colell, “A simple adaptive procedure leading to correlated equilibrium”.

Correlated Nash equilibrium

- If players can engage in preplay communication, then go in separate rooms and choose their strategy independently
  - Might gain if they can build a signaling device → send signals to the separate rooms.
- Example of game with correlated equilibria

	L	R
U	5,1	0,0
D	4,4	1,5

  - Three Nash equilibria: (U,L), (D,R), and a mixed strategy eq. with equal probability on each pure strategy: payoff 2.5 for each player

Correlated Nash equilibria: cont

- If they can jointly observe a random variable, e.g. a coin flip
  - Player 1: U if heads, D if tails
  - Player 2: L if heads, R if tails

  → Payoff: (3,3)
  - More general: players can obtain any payoff vector in the convex hull of the set of Nash equilibrium payoffs
  - Cannot obtain any payoff outside the convex hull of the set of Nash equilibrium payoffs
- Can gain further if players receive different signals, but correlated
  - Build a signaling device that sends different, but correlated, signals to each of them
  - Device has three equally likely states: A, B, C
  - If A – player 1 completely informed, but it cannot distinguish between B and C
  - Player 2 – informed when C, cannot distinguish from A and B

Correlated Nash eq. – cont.

- A Nash eq. for the transformed game:
  - Player 1 plays U when A and D ow
  - Player 2: R when C, and L ow
  - Eq: (U,L), (D,L) and (D,R), occur with probability 1/3, payoff 3.33
  - Payoff outside the convex hull of the set of Nash equilibrium payoffs
- Note: signaling device may be interpreted as a recommendation on how to play
- Definition: A correlated eq. is any probability distribution p(.) over the pure strategies S1 x S2 x… Si, s.t. for every player i and every si, with p(si)>0,

$$\sum_{s_{-i} \in S_{-i}} p(s_{-i} \mid s_i) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} p(s_{-i} \mid s_i) u_i(s_i^{'}, s_{-i}) \quad \forall \ \ s_{i'} \in S_i$$

- Player I should not be able to gain by disobeying the recommendation to play si, if every other player obeys its recommendation

Types of games

Dummy game

- Unilateral deviations produce no change in the payoff of the deviating player.

a,b	c,b
a,d	c,d

- All pure strategy profiles are Nash eq.

$$u_i(s_i, s_{-i}) = u_i(s'_i, s_{-i})$$

Types of games: coordination games

Identical Interest Game

- For any choice of pure strategies, same payoffs for the players

a,a	b,b
c,c	d,d

$$u_i(s) = u_j(s), \ \ \forall\, i, j \in N, \ \forall s \in S$$

- There exists some function *V*: *S* $\rightarrow \Re$ such that

$$u_i(s) = V(s) \ \ \forall i \in N, \ \forall\, s \in S$$

- All Maximizers of *V* are Nash eq.
- At least one Nash eq. must be Pareto Efficient

Types of games: supermodular games

- Games in which each player’s marginal utility of increasing its strategy rises with increases in its rivals’ strategies
- Super-modular games have pure strategy Nash equilibria

- To define supermodularity we need
  - Order structure on strategy spaces
  - Weak continuity requirements on payoffs
  - Supermodularity requirement

Order relation

- Let x and y denote two vectors in $\mathbf{R}^K$
- Let $x \geq y$ if $x_k \geq y_k$ for all $k = 1,2,\dots,K$
- Let $x > y$ if $x \geq y$ and there exists some k such that $x_k > y_k$
- If a vector dominates another in one component, but is dominated in another component, vectors cannot be compared

Example:

$x = (2,1,5,3)$, $y = (3,5,7,3)$ $y \geq x$ $y > x$

$x = (2,0,8,1)$, $y = (1,5,9,1)$ Cannot be compared

“Meet” and “join” relations

“meet of *x* and *y*” $x \wedge y \equiv (\min\left\{x_1, y_1\right\}, \min\left\{x_2, y_2\right\}, \ldots, \min\left\{x_K, y_K\right\})$

“join of *x* and *y*” $x \vee y \equiv \{\max\left(x_1, y_1\right), \max\left(x_2, y_2\right), \ldots, \max\left(x_K, y_K\right)\}$

S is a sublattice of $\mathbf{R}^{m_i}$ if *s*, *s** $\in$ *S* then *s* $\wedge$ *s** $\in$ *S* and *s* $\vee$ *s** $\in$ *S.*

Sublattice property – Every bounded sublattice has a greatest and a least element.

Increasing Differences Property

Definition

$u_i(s_i, s_{-i})$ has increasing differences in (s_i, s_{-i}) if, for all

$s_i, \tilde{s}_i \in S_i$ and $s_{-i}, \tilde{s}_{-i} \in S_{-i}$ such that $s_i \geq \tilde{s}_i$ and $s_{-i} \geq \tilde{s}_{-i}$

$$u_i\left(s_i, s_{-i}\right) - u_i\left(\tilde{s}_i, s_{-i}\right) \geq u_i\left(s_i, \tilde{s}_{-i}\right) - u_i\left(\tilde{s}_i, \tilde{s}_{-i}\right)$$

Intuitive explanation: an increase in the strategies of *i*’s rivals Increases the value of playing a high strategy for player *i*.

Supermodular Games

Definition:

$u_i(s_i, s_{-i})$ is supermodular in s_i if for each s_{-i}

$$u_i\left(s_i, s_{-i}\right) + u_i\left(\tilde{s}_i, s_{-i}\right) \le u_i\left(s_i \wedge \tilde{s}_i, s_{-i}\right) + u_i\left(s_i \vee \tilde{s}_i, s_{-i}\right) \ \forall s_i, \tilde{s}_i \in S_i$$

The above relation is satisfied with equality if S_i is single-dimensional

Definition:

A game is supermodular if the following conditions are met for all i

S_i is a sublattice of $\mathbf{R}^m$

u_i has increasing differences in (s_i, s_{-i})

u_i is supermodular in s_i

Continuous payoff functions

- If u_i is twice continuously differentiable, u_i is a supermodular game iff, for any s_k and s_l (components of s)

$$\boxed{\frac{\partial^2 u_i}{\partial s_l \partial s_k} \geq 0}$$

- Example: Bertrand game
  - Oligopoly with demand function:

$$D_i(p_i, p_{-i}) = a_i - b_i p_i + \sum_{j \neq i} d_{ij} p_j, \ \ b_i > 0, \ \ d_{ij} > 0$$

  - And utility:

$$u_i(p_i, p_{-i}) = (p_i - c_i) D_i(p_i, p_{-i})$$

  - Supermodular game: $\dfrac{\partial^2 u_i}{\partial p_i \partial p_j} > 0, \quad \forall i, j \neq i$

Nash eq. for supermodular games

- (Topkis) A supermodular game for which each S_i is compact and each u_i is upper semi-continuous in s_i for each s_{-I}, then the set of pure strategy Nash eq. is non-empty and contains greatest and least elements $\overline{s}, \underline{s}$,respectively

Note on upper semi-continuity:

A function *f:X* → *R*

- upper semi-continuous if, for every a in R, the preimage of [a,∞) is closed.
- lower semi-continuous if, for every a in R, the preimage of (-∞,a] is closed

f:X → *R, the image of x is f(x). The preimage of y is* $f^{-1}(y) = \{x \mid f(x) = y\}$ *, for all x whose image is y*

Best response for supermodular games

- (Milgrom and Roberts) A best response dynamic for supermodular game with compact action spaces and upper semi continuous objective functions converges to a region bounded by the greatest and least elements in the set of Nash eq.

- If the Nash eq. is unique, then the best response dynamic converges to the Nash eq..

Potential games

Exact potential game:

- A game for which you can construct a single-dimensional function

  P: $S \rightarrow \Re$ whose change in value is exactly equal to the utility change in value of the deviating player.

$$P(s_i, s_{-i}) - P(s'_i, s_{-i}) = u(s_i, s_{-i}) - u(s'_i, s_{-i})$$

Coordination games: potential games
Combination of coordination and dummy games: potential games
Most of congestion games: potential games

- A game $G = \langle N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ is an exact potential game *iff* there exist functions $\{c_i\}_{i \in N}$ and $\{d_i\}_{i \in N}$ such that
  - $u_i = c_i + d_i$
  - $\langle N, \{S_i\}_{i \in N}, \{c_i\}_{i \in N} \rangle$ is a coordination game
  - $\langle N, \{S_i\}_{i \in N}, \{d_i\}_{i \in N} \rangle$ is a dummy game

Nash equilibrium

- The Nash equilibria of an exact potential game are the same as the ones for its constituent coordination game.

- Proof sketch:
  - unilateral deviation for a dummy game yields the same payoff → adding a dummy game *D* to any other game will preserve the Nash eq.
  - Any potential game can be decomposed as the sum of a coordination game and a dummy game
  - Nash eq. for the potential game must be the same as for the constituent coordination game

Nash eq. More properties

- For an exact potential game, the maximizers of the potential function are Nash eq. for the game.
- For a finite exact potential game, i.e., finite strategy space, finite player set, the game has at least one pure-strategy Nash equilibrium.
- All repeated games where each stage is the same finite Exact Potential Game (EPG) and all players are myopic, converge to the Nash equilibria of the stage game, if play follows a better response dynamic.
  - Definition: Better response dynamic

    At each stage, one player $i \in N$ is permitted to deviate from s_i to some randomly selected action $s'_i \in S_i$ iff it is an improvement deviation (the action improves utility).
- All repeated myopic games with the stage game an EPG, converge to the Nash eq. of the stage game, if play follows a best response dynamic.

How to recognize potential games?

- Find potential function
  - Using def. $P(s_i, s_{-i}) - P(s'_i, s_{-i}) = u(s_i, s_{-i}) - u(s'_i, s_{-i})$
  - For continuous payoff:

$$\frac{\partial P}{\partial s_i} = \frac{\partial u_i}{\partial s_i}, \quad \forall\, i \in N$$

$$\frac{\partial^2 P}{\partial s_i \partial s_j} = \frac{\partial^2 u_i}{\partial s_i \partial s_j} = \frac{\partial^2 u_j}{\partial s_i \partial s_j}, \quad \forall \mathrm{i, j} \in \mathrm{N}$$

How to recognize potential games?

Bilateral Symmetric Interaction Game

Strategic form game for which a player’s objective function is a sum of bilateral symmetric interaction (BSI) terms.
A BSI term is defined as

$$w_{ij} : S_i \times S_j \rightarrow \mathcal{R}\,, \text{ s.t. } w_{ij}(s_i, s_j) = w_{ji}(s_j, s_i) \;\; \forall (s_i, s_j) \in S_i \times S_j$$

The objective function: $u_i(s) = \sum_{j \neq i} w_{ij}\left(s_i, s_j\right) - h_i(a_i)$

Potential function for this game can be defined as:

$$\sum_{i \in N} \sum_{j=1}^{i-1} w_{ij}(s_i, s_j) - \sum_{i \in N} h_i(s_i)$$

How to recognize potential games?

- Coordination game
- Coordination + dummy games
- Congestion games

Congestion games

- Players use several facilities (common resource)
- The players' utility derived from the use of a facility depends *only* on the number of users sharing the facility
- The payoff of a player = sum of the benefits associated with each facility in his strategic choice, given the choices of other players

- A class of congestion games – introduced by Rosenthal, 1973 – exact potential games

- *Every exact potential game is isomorphic to a congestion game*
- *Each coordination game is isomorphic to a congestion game*
- *Each dummy game is isomorphic to a congestion game*

Congestion game model

A congestion model: <*N, F,* $(X_i)_{i \in N}$, $(w_f)_{f \in F}$>, where

- *N* = nonempty, finite set of players
- *F* = nonempty, finite set of facilities
- For each player $i \in N$, its collection of pure strategies X_i is a nonempty, finite family of subsets of *F*
- For each facility $f \in F$, $w_f : \{1,\ldots,n\} \rightarrow \mathcal{R}$ is the benefit of facility *f*, with $w_f(r)$, $r \in \{1,..,n\}$ = *the benefits of each of the users of facility f, if there is a total of r users*

A congestion game G= <*N,* $(X_i)_{i \in N}$, $(u_i)_{i \in N}$>

$$u_i : X \rightarrow \mathcal{R} \quad u_i(x) = \sum_{f \in x_i} w_f(n_f(x))$$

$$n_f(x) = \left|\{i \in N : f \in x_i\}\right|$$

Congestion game – exact potential game

- Prop. *Let <N, F, $(X_i)_{i \in N}$, $(w_f)_{f \in F}$> be a congestion model and G its congestion game. Then G is an exact potential game. A potential function is given by P: X → $\mathcal{R}$*, defined for all $x = (x_i)_{i \in N} \in X$ as

$$P(x) = \sum_{f \in \bigcup_{i \in N} x_i} \sum_{l=1}^{n_f(x)} w_f(l)$$

- Since *X* is finite, the game has a Nash equilibrium in pure strategies

Coordination game – isomorphic to congestion game

- What is isomorphic?
- *Let G= <N, $(X_i)_{i\in N}$, $(u_i)_{i\in N}$> and H = <N, $(Y_i)_{i\in N}$, $(v_i)_{i\in N}$>, be two strategic games with identical player set N. G and H are isomorphic, if*

$$\forall i \in N, \quad \exists\, \phi_{\mathrm{i}} : X_i \rightarrow Y_i \text{ , s.t.}$$

$$u_i(x_1,x_2,\ldots,x_n) = v_i(\varphi_1(x_1),\varphi_2(x_2),\ldots,\varphi_n(x_n)) \quad ,\forall \ (x_1,x_2,\ldots,x_n) \in X$$

- *A congestion game for which the facilities have non-zero benefits only if all players use it as part of their strategy → isomorphic to a coordination game*

Coordination game – isomorphic to congestion game

Theorem: Each coordination game is isomorphic to a congestion game.

A simple example:

0,0	1,1
2,2	3,3

Coordination game

A	B
C	D

	{A,C}	**{B,D}**
{A,B}	0,0	1,1
{C,D}	2,2	3,3

Isomorphic congestion game

Dummy games – isomorphic to congestion games

- *A congestion game for which the benefits for a facility are non-zero only if it used by a single player → isomorphic with a dummy game*

Theorem: Each dummy game is isomorphic to a congestion game

Simple example:

0,2	1,2
0,3	1,3

Dummy game

α,γ	β,γ
α,δ	β,δ

	{β,γ,δ}	{α,γ,δ}
{α,β,δ}	0,0	1,1
{α,β,γ}	2,2	3,3

Isomorphic congestion game

Congestion games and potential functions

- *Theorem: Every exact potential game is isomorphic to a congestion game.*

- *All congestion games → potential games?*
  - *Not all classes of congestion games admit a potential function*
  - *Existence of a pure Nash equilibrium strategy is proved based on specific properties of the congestion game*

Strategic interaction properties for congestion games

- *(P1) There exists a finite set F, s.t.* X_i *= F for all players i* $\in$*N*
  - *F = facility set. Strategy for players: choose an element of F.*

- *(P2) Independence of irrelevant choices: For each player i* $\in$*N, and each strategy profile x, the utility of i will not be altered if the set of players that choose the same facility as player i is not modified*

$$\forall x \in X,\ i, j \in N : \text{ if } x_i \neq x_j, \text{and } x'_j \in X_j, x_i \neq x'_j \Rightarrow u_i\left(x_j, x_{-j}\right) = u_i\left(x'_j, x_{-j}\right)$$

- *(P3) Anonymity condition: The payoff of player i depends on the number of players choosing the facilities, rather than on their identity.*

$$\forall i \in N,\ \forall x, y \in X,\ \ x_i = y_i : \text{ if } \ n_f(x) = n_f(y),\ \forall f \in F,\ u_i(x) = u_i(y)$$

More properties…

- *(P4) Partial Rivalry: each player I would not regret that other players, choosing the same facility, would select another one.*

$$\forall i \in N,\ \forall x \in X,\ \ \forall j \neq i,\ \text{s.t. } x_j = x_i,$$

$$\text{and each}: \ x'_j \neq x_i: \ \ u_i(x_j, x_{-j}) \leq u_i(x'_j, x_{-j})$$

- *Theorem: Each game <N, $(X_i)_{i \in N}$, $(u_i)_{i \in N}$> satisfying (P1), (P2), (P3) and (P4) possesses a pure-strategy Nash equilibrium.*

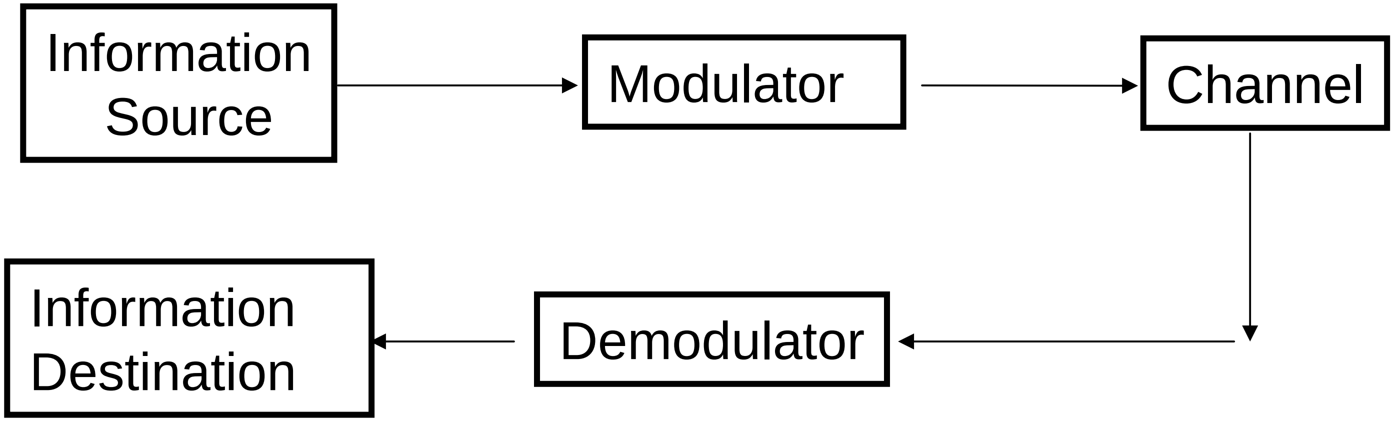
- *(P5) For all strategy profiles x,y$\in$X, and all players i,j $\in$ N: if $x_i = y_j$ =f and $n_f(x)=n_f(y)$, then $u_i(x)=u_j(y)$.*

- *Theorem: Each game satisfying (P1) and (P5) is an exact potential game.*

References for congestion games

- M. Voorneveld, P. Borm, F. Megen, S. Tijs, G. Facchini, “Congestion Games and Potentials Reconsidered”, International Game Theory Review, vol 1, pp. 283-299, 1999 (http://greywww.kub.nl:2080/greyfiles/center/1999/doc/98.pdf).
- R.W. Rosenthal, “A class of games possessing pure-strategy Nash equilibria”, International Journal of Game Theory, vol. 2, pp.65-67, 1973.

Simple model for wireless transmission



Information Source

Modulator

Channel

Demodulator

Information Destination

Assume information source is digital: generates a string of bits that must be transmitted using electromagnetic waves (no wires)

- modulates a carrier
- sinusoidal signals – suitable carriers, $A\sin(2\pi f t + \theta)$ characterized by
  - amplitude: amplitude modulation
  - frequency: frequency modulation
  - phase: phase modulation

Example: BPSK:

$$s_0(t) = A\sin(2\pi f_c t + \pi) = -A\sin(2\pi f_c t), \ 0 \le t \le T$$

$$s_1(t) = A\sin(2\pi f_c t) \quad , \quad 0 \le t \le T$$

Modulation examples – Cont.

- QPSK: modulate both the sine and cosine (the quadrature) carrier

$$A_1 \sin(2\pi f t + \theta) + A_2 \cos(2\pi f t + \theta)$$

  - Better spectral efficiency:

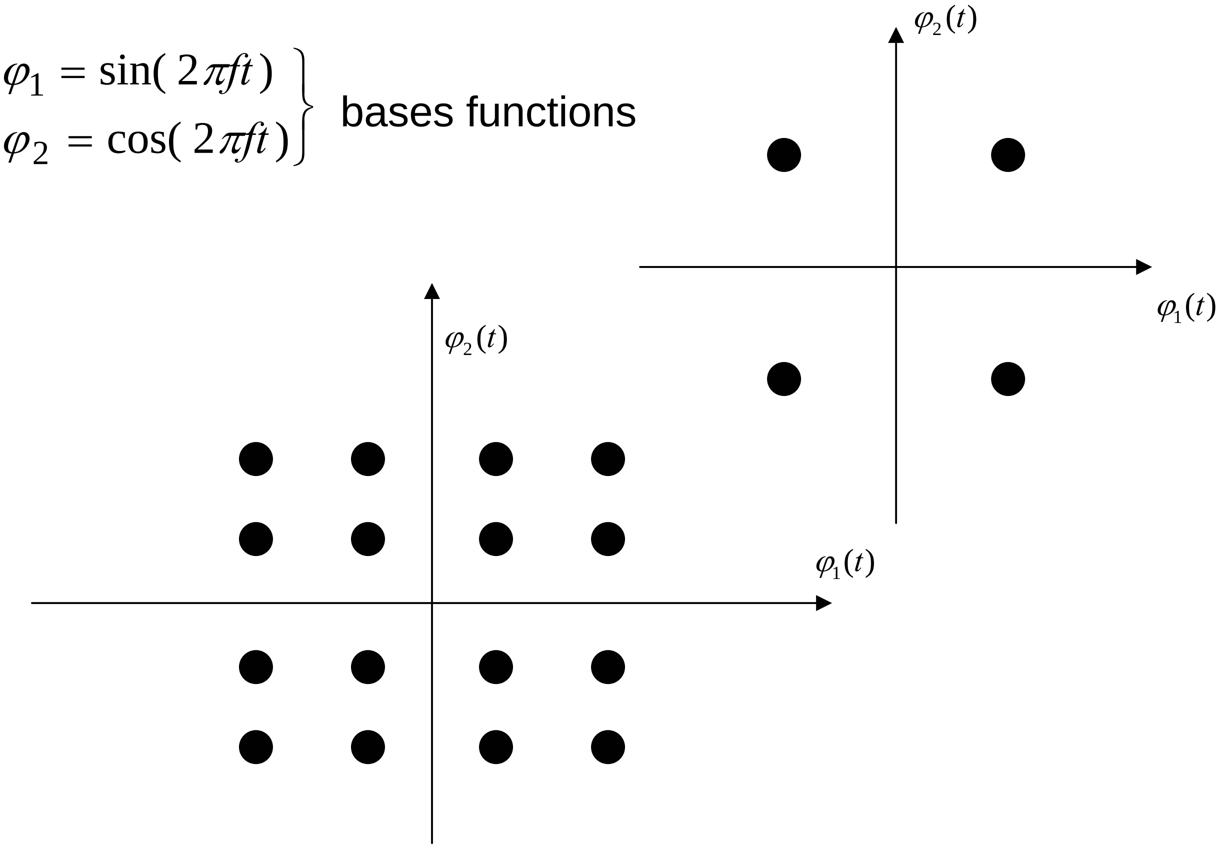
$$\text{Spectral Efficiency} = \frac{\text{Bit rate}}{\text{Transmission Bandwidth}} \quad (bps/Hz)$$

- Can you improve further the spectral efficiency?
  - M-ary modulation
    - Example M - QAM $A_1, A_2 = \pm 1, \pm 3, ..., \pm \sqrt{M} - 1$
      - $\log_2 M$ bits encoded into one symbol
  - Large M – higher rates!!!
    - Question: can we get unlimited high rates for a given bandwidth by increasing M ?
      - **NO: we have to be able to distinguish between the received symbols**

Signal Constellation and Detection

- 4-QAM: $A_1 \sin(2\pi f t + \theta) + A_2 \cos(2\pi f t + \theta) = A_1\varphi_1(t) + A_2\varphi_2(t)$

$$\left.\begin{cases} A_1 = \pm 1 & \varphi_1 = \sin(2\pi f t) \\ A_2 = \pm 1 & \varphi_2 = \cos(2\pi f t) \end{cases}\right\}$$ bases functions



- 16 –QAM

$$\begin{cases} A_1 = \pm 1, \pm 3 \\ A_2 = \pm 1, \pm 3 \end{cases}$$

  - Higher constellation, less room for errors
  - Problem: channel introduces noise, fading, distortions

Physical Channel

- Higher-order (M-ary) → increased spectral efficiency
- Rate of reliable data transmission

→ limited by impairments due to physical properties of the channel:

- noise (receiver & background)
- path losses (spatial diffusion & shadowing)
- multipath (fading & dispersion)
- interference (multiple-access & co-channel)
- dynamism (mobility, random-access & bursty traffic)
- limited transmitter power

Noise

- Noise present in all communication systems.
  - White Gaussian noise:
    - spectrum constant for all frequencies
    - pdf is Gaussian
  - Key parameter of noise:
    - zero mean
    - spectral height No/2 = σ^2(variance of the noise)

  - Key performance parameter when no interference is present:
    - SNR = Eb/No (signal-to-noise ratio) (Eb = received energy per bit)
    - Determines BER (bit error rate)
      - Different BER for different modulation types

Propagation effects

- Two basic types of propagation effects:
  - Large-scale (spatial diffusion & shadow fading)
  - Small-scale (multipath fading)

- Propagation in free space: ignores any interactions
  - Antenna radiates a sine wave with the carrier frequency

$$f = \frac{c}{\lambda} \qquad c = 3\times 10^{8}$$ speed of light

  - Friis free space equation:

$$\boxed{P_r = P_t\left(\frac{\lambda}{4\pi r}\right)^2 g_t g_r}$$

P_t = transmit power
P_r = received power
g_t, g_r = transmit/receive antenna gains
r = distance between the antennas

Distance based attenuation

Propagation along the earth’s surface: 2-ray model

- Flat earth assumption
- Ground wave reflected
  - Delay
  - Phase shift
  - Attenuation

$$\boxed{P_r = P_t \left(\frac{h_t h_r}{r^2} \right)^2 g_t g_r}$$

Approximation path loss model with n = path loss coefficient:

$$\boxed{P_r = P_t g_t g_r \frac{const}{r^n}}$$

-for omnidirectional antennas: $g_t = g_r = 1$

Large and small time scale fading

Fading effects - different at different time scales

- the instantaneous signal envelope (short time scales (ms)) is
  - Rayleigh distributed (NLOS)
  - Rice distributed (LOS)
- the mean value of the Rayleigh (or Rice) distribution can be considered a constant for the shorter time scales, but in fact it is a random variable with a lognormal distribution (large time scales (seconds))
  - caused by the changes in scenery (occur on a larger time scale)
- the mean of the Lognormal distribution varies with the distance from the transmitter according to the path loss law
  - If the mobile moves away or towards the transmitter (e.g. base station) the received signal will also vary in time, according to the appropriate power law loss model (e.g. free space: decreases proportional with the square of the distance, etc.)

Multiuser communication: sharing the spectrum

Multiple users' transmissions interfere with each other

- Users need to be separated
  - Frequency (FDMA)
  - Time (TDMA)
  - Using different codes (CDMA)
  - In space (spatial separation in cellular and ad hoc networks)
- Performance measure: BER (Bit error rate)
  - characteristic for the type of service (e.g. req. voice BER $\cong 10^{-2}$).
  - Can be mapped into a SIR (SINR) (Signal to Interference plus noise ratio) requirement

Network architecture: Users may all transmit to the same access point (cellular, wireless LAN), or they can use peer-to-peer communication (ad hoc network)

- no wires → soft link concept – depends on the reception quality

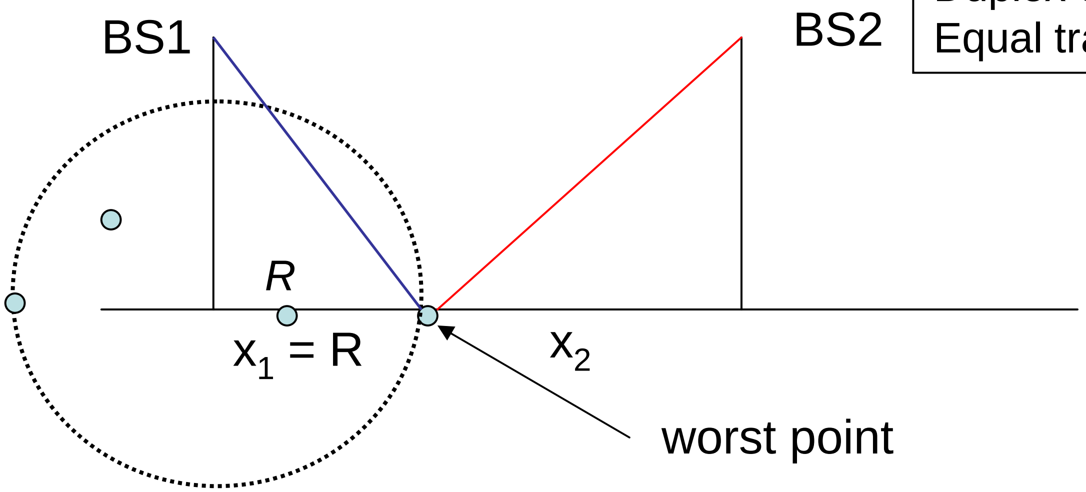
Interference example: channel reuse

- -How should we design a cellular system
  - -One base station (BS), high power and large coverage?
  - -Split into cells to accommodate a larger density of users?

Example: 2 BS use the same channels and are situated at distance D
Question: how should we choose D/R (R is the coverage radius for one cell,
Such that all users meet their target SIR (signal-to-interference ratio)
- - SIR maps the bit error rate performance (BER)

- Example – continuation

Assume noise is 0
Channel impairment is determined by other users using the same channel
Duplex communication
Equal transmission powers



$D = x_1+x_2$
T = target SIR in dB

$$\left.\begin{aligned}\overline{P_S} &= \frac{const}{{x_1}^n} P_t \\ \overline{P_I} &= \frac{const}{{x_2}^n} P_t\end{aligned}\right\} \Rightarrow \overline{SIR} = \frac{\overline{P_S}}{\overline{P_I}} = \left(\frac{x_2}{x_1}\right)^n = \left(\frac{D-R}{R}\right)^n$$

$$\overline{SIR}_{dB} = 10 \log_{10} \overline{SIR} = 10\, n \log_{10}\left(\frac{x_2}{x_1}\right) \geq T$$

- Example – continuation

$$\frac{x_2}{x_1} \geq 10^{\frac{T}{10n}} \Rightarrow \frac{D}{R} \geq 1 + 10^{\frac{T}{10n}}$$ → want small or big number?

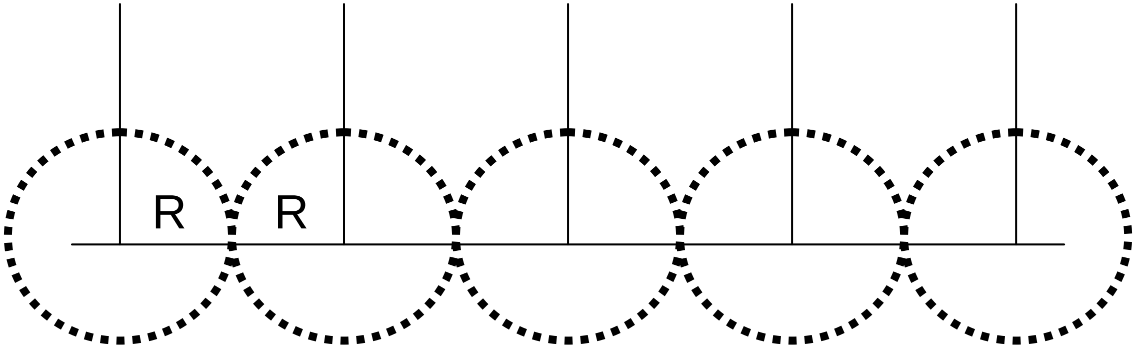
From cellular efficiency point of view -> want small numbers

The effective number of channels/cell = total number of channels/ cell reuse factor (N)

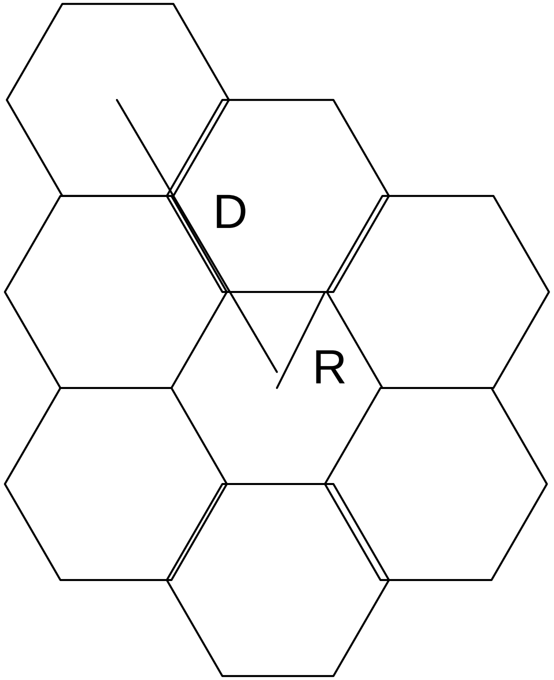
How to determine N for given SIR requirement ?

For base stations situated in a straight line, it can be shown that

$$N = \frac{1}{2}\frac{D}{R}$$



- For hexagonal cells



$$N = \frac{1}{3}\left(\frac{D}{R}\right)^2$$

For CDMA systems: N=1

We can define the cellular efficiency:

$$\eta = \frac{B_S}{B_C N} \text{ channels/cell}$$

B_S = spectrum allocated to the cellular system
B_C = bandwidth/channel
N = channel reuse

Sharing the spectrum: CDMA

- Basic CDMA principle: all users transmit simultaneously using the same frequency band and are characterized by different signature sequences codes $\boldsymbol{s}_i$, $i=1,2,\ldots, K$ (K = number of users)
  - Signature codes can be selected to be orthogonal: users are completely separated from each other $\mathbf{s}_i^T \mathbf{s}_j = 0;\ i \neq j\ \ \mathbf{s}_i^T \mathbf{s}_i = 1$

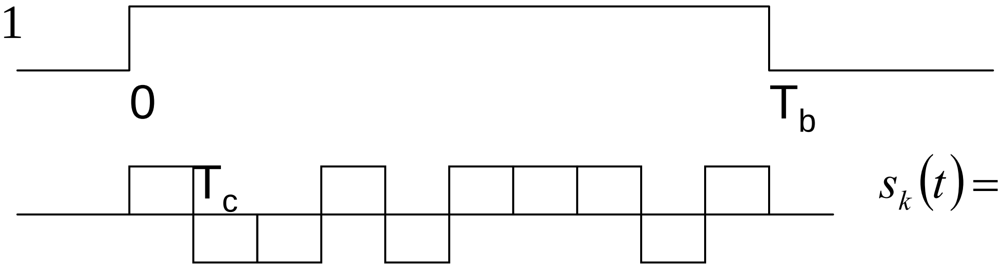
    Disadvantages:

    - number of users that can be supported in the system is limited by the number of orthogonal codes
      - If length of code is N, $\max\{ K \} = N$
    - Orthogonality cannot be maintained for asynchronous transmission
  - Codes can also be selected non-orthogonal but with small cross-correlations
    - Random codes – all entries are -1 or 1 with equal probability (coin flips)
    - Pseudo-random codes (IS-95 cellular CDMA) – m-sequences
      - very long sequences cyclically repeated (generated by linear shift registers) – appear as random
      - different statistical properties than random codes

Simple single user CDMA system

$b_k(t) = b_k \times p_T(t)$ - bit waveform

$b_k = +1$



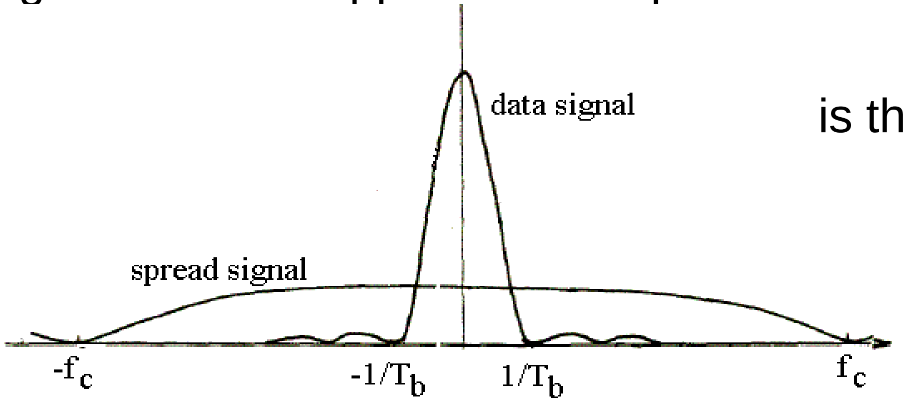
$N = \frac{T_b}{T_c}$ = spreading gain

$$s_k(t) = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} s_{ij} p_{T_c}\left(t - (j-1)T_c\right)$$

- signature sequence waveform

-signature sequence code: $\mathbf{s} = [+1,-1,-1,+1,-1,+1,+1,+1,-1,+1]$

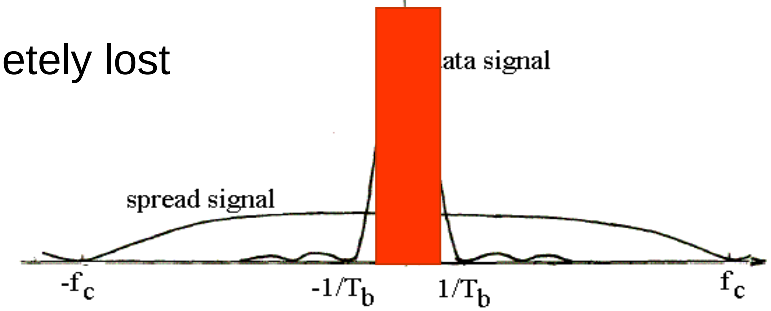
- Multiply them together: what happens to the spectrum?



is this good or bad?

Properties of spread spectrum

- At a first glance: bad – uses more bandwidth for the same transmission rate
- Very important advantages:
  - Resistant to frequency selective fading
    - A deep fade affects only partially the signal on that particular frequency – signal can still be recovered
    - Without spreading – the signal is completely lost
  - Exploits multipath: rake receiver
  - Low probability of intercept
    - Low signal level – noise like
      - Hard to eavesdrop
      - Creates reduced interference to other users
  - Resistance to jamming and interference
    - Narrow band jamming and interference affects only partially the signal
  - The last two properties are particularly attractive for unlicensed bands
  - Because of its resistance to interference: can have frequency reuse 1 - big capacity advantage

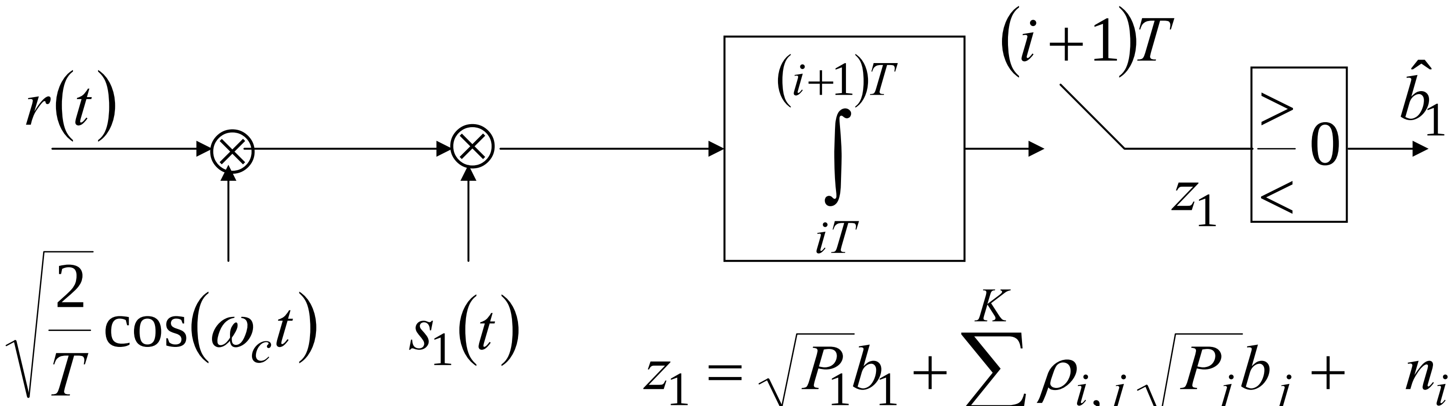


Multiple users

- Every user has a different signature sequence
- The received signal

$$r(t) = \sum_{k=1}^{K} A_k b_k s_k(t) + n(t) \qquad A_k = \sqrt{P_k}$$

- To detect signal 1: matched filter receiver



$$z_1 = \sqrt{P_1} b_1 + \underbrace{\sum_{j \neq i}^{K} \rho_{i,j} \sqrt{P_j} b_j}_{MAI} + \underbrace{n_i}_{AWGN}$$

MAI = multi-access interference

Performance and optimality

- Performance depends on
  - Powers: implement power control
  - Cross-correlations:

$$\rho_{i,j} = \int_0^T s_i(t) s_j(t) dt = \sum_{n=1}^{N} s_{in} s_{jn} = \mathbf{s}_i^T \mathbf{s}_j$$

  - For random sequences $\begin{cases} E[\rho_{ij}] = 0 \\ \mathrm{var}[\rho_{ij}] = \frac{1}{N} \end{cases}$
  - Matched filter: optimal for Gaussian noise
    - Assumption: central limit theorem – interference is Gaussian

$$\begin{cases} E[I] = 0 \\ \mathrm{var}[I] = P(K-1)\frac{1}{N} \end{cases}$$ = interference power

Probability of error

BPSK/QPSK: $P_e = Q\left(\sqrt{SIR}\right)$

If K is large – neglect the noise contribution

All received powers equal

$$P_e = Q\left(\frac{\sqrt{p}}{\sqrt{\frac{p}{N}(K-1)}}\right) = Q\left(\sqrt{\frac{N}{K-1}}\right)$$

SIR – key measure for performance – determines capacity: soft capacity

Note: a K user asynchronous system ~ (2K-1) synchronous system (virtual users)
for asynchronous systems:

$$P_e = Q\left(\sqrt{\frac{3N}{K-1}}\right)$$

Resource allocation for wireless networks

- Transmission power assignment
- Channel allocation (frequency, time, codes)
- Rate assignment
- Route selection

Static or dynamic ?

- Mobility, burstiness, fading, etc → interference conditions change in time

Centralized or distributed ?

- centralized: works for downlink in cellular systems
- distributed:
  - uplink in cellular – centralized would require a lot of signaling overhead
  - ad hoc networks – self-organizing networks, no infrastructure

Game theory for resource allocation

- Users interact with each other by creating interference
- Game theoretic formulation:
  - N users, choose actions:
    - Transmitted power level
    - Channel (frequency, time, code – waveform selection)
    - Transmission rate
    - Transmission route (for peer-to-peer connections in multi-hop ad hoc networks)
  - Rewards (utilities) – associated with transmission quality
    - Achieved BER (SIR)
    - Energy expenditure
    - Transmission delay
    - Throughput

Game theory for resource allocation

- Defining the interactions between users, their strategies and their utilities → different game formulations
- Study the performance of these games
  - Convergence
    - Existence and uniqueness of Nash equilibrium
    - Conditions for convergence
  - Efficiency (optimality)
    - Best possible performance?
    - Pareto efficiency?
  - Fairness
    - Are resources shared equitably between users?

Reference for introduction to wireless communications

- “Wireless Communications: Principles and Practice”, T.S. Rappaport, December 2001, Prentice Hall

Power control for wireless systems

- Recall our wireless design example last class:
  - Physical layer performance measure: bit error rate (BER)
  - Based on the used modulation scheme: BER target can be mapped into an SIR (signal to interference ratio) target
  - Reliable communication → meet target BER/SIR
    - **How can you achieve this?**
      - **WIRELESS SYSTEMS: INTERFERENCE LIMITTED**
      - **Dynamically adjust to the current interference pattern (level):**
        - » **Change powers**
        - » **Change transmission rate**
        - » **Waveform adaptation**
        - » **MAC: schedule transmission**
        - » **Routes: affect interference distribution in an ad hoc network**

Power control

Select your power level that you exactly meet your target SIR, γ_0

battery drain

- If SIR > γ_0, use too much power

interference with others

- If SIR < γ_0 , packets cannot be received correctly →
  → retransmissions – energy inefficient

Power Control cont.

- Assume that: ***Q*** transmitters use the same channel C_0

  They have power: $P = (p_1, p_2, \dots, p_Q)^T$

  p_i the power at the i^{th} transmitter

  $i = 1,2,\dots,Q$

- The expression for SIR at receiver i is

$$\boxed{SIR_i = \frac{g_{ii} p_i}{\sum_{j=1, j \neq i}^{Q} g_{ij} p_j + n_i}}$$

g_{ij} - link gain

n_i - noise power at receiver i

Power Control cont.

- Transmitter i is supported if :

$$SIR_i \geq \gamma_0 \qquad \gamma_0 \text{ - target SIR}$$

=> $$p_i \geq \gamma_0 \left(\sum_{j=1,\, j \neq i}^{Q} \frac{g_{ij}}{g_{ii}} p_j + \frac{n_i}{g_{ii}} \right) (*)$$

power to select if all other powers are kept fixed

-Denote $\frac{g_{ij}}{g_{ii}} = h_{ij}$ $\frac{n_i}{g_{ii}} = \eta_i$

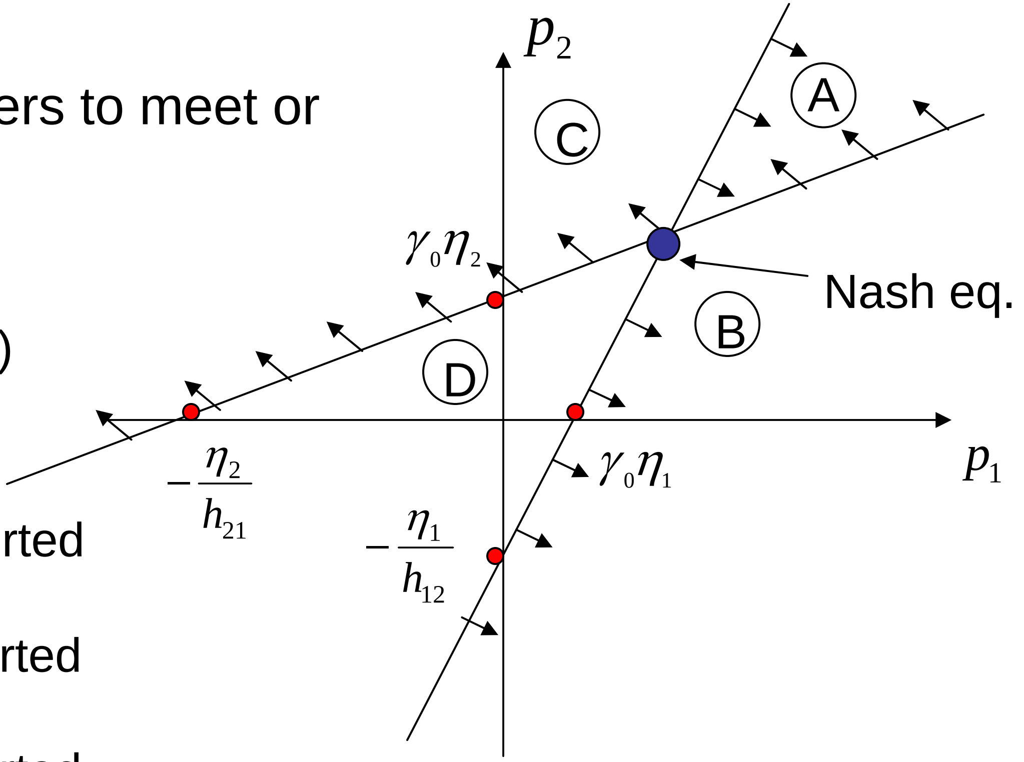
- In a system $(*)$ has to be hold for all $i = 1,2,\ldots,Q$

Game: strategy – power
utility - SIR

Simple 2 user example

- Users adjust their powers to meet or exceed target SIR:

$$\begin{cases} p_1 \geq \gamma_0\left(h_{12}\,p_2 + \eta_1\right) \\ p_2 \geq \gamma_0\left(h_{21}\,p_1 + \eta_2\right) \end{cases} \quad (*)$$



Ⓐ - both users can be supported

Ⓑ - only user1 can be supported

Ⓒ - only user2 can be supported

Ⓓ - none can be supported

Reaction functions

Minimum power solution: achieved for equality in (*)

$$\begin{cases} p_1 = \gamma_0\left(h_{12}\,p_2 + \eta_1\right) \\ p_2 = \gamma_0\left(h_{21}\,p_1 + \eta_2\right) \end{cases}$$

Nash equilibrium and minimum power solution

- **Game theoretic solution:**

- **Nash equilibrium**

- Existence?
- Uniqueness?
- Pareto efficiency?

⟷

- **Classic approach**

- **Minimum power solution**

Feasibility condition for power control

- Power efficiency???

Power Control Feasibility

- How many users can you support to maximize capacity, while maintaining SIR requirement?

- Feasibility conditions:

  For Q users:

$$\boxed{(I - H)P \geq \eta}$$

$$H_{Q\times Q} \longrightarrow H_{i\,j} = (h_{i\,j}) \qquad h_{i\,j} = \begin{cases} \gamma_0 \dfrac{g_{ij}}{g_{ii}} & i \neq j \\ 0 & i = j \end{cases}$$

$$\eta = (\eta_1, \eta_2 \ldots\ldots \eta_Q)^T$$

Power Control Feasibility cont.

- Def. The target SIR γ_0 is said to achievable, if there exists a non-negative power vector so that $(*)$ holds for all i .

- The target SIR γ_0 is achievable if the dominant (largest) eigenvalue of matrix H, ($\rho(H)$) is less or equal to one.

$\rho(H)=1 \longrightarrow$ γ_0 is achievable only when noise is zero

$\boxed{\rho(H)<1} \longrightarrow$ Power control feasibility condition

Distributed power control: standard interference function

- To support the design of various power control algorithms, a general framework for proving the convergence of iterative distributed power control algorithms was proposed.

  -**Standard Interference Function (Yates '95)**

  Power control algorithm: iterative

$$P^{(n+1)} = I(P^{(n)})$$

  I : Interference function – define the power vector of the next iteration

- Def. Assuming positive receiver noise, an interference function ***I*** is called standard if it satisfies all non-negative power vectors:

  1) Positivity $I(P) > 0$

  2) Monotonicity $P \geq P' => I(P) \geq I(P')$

  3) Scalability $\forall \alpha > 1, \ \alpha \cdot I(P) > I(\alpha P)$

  ← Component wise

$$\alpha \left(\sum_{j=1,\, j \neq i}^{Q} \frac{g_{ij}}{g_{ii}} p_j + \frac{n_i}{g_{ii}} \right) \qquad \left(\sum_{j=1,\, j \neq i}^{Q} \frac{g_{ij}}{g_{ii}} \alpha p_j + \frac{n_i}{g_{ii}} \right)$$

Standard Interference Function cont.

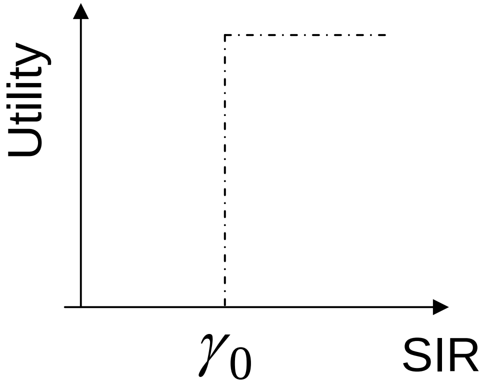
- Proposition: If the system is feasible, the sequence of power vectors from the standard interference function will converge to the minimum power solution vector P^* , starting with any non-negative power vector, and the rate of convergence is geometric.

- Observation on convergence:
  - Starting with all zero power vector, the sequence $P(n) = I(P(n-1))$ is monotonically increasing.
  - If $P(0) = I(P(0))$, then the sequence $P(n) = I(P(n-1))$ is monotonically decreasing.
  - Staring with any initial power vector P , the sequence $P(n) = I(P(n-1))$ converges geometrically to the fixed point P^*.

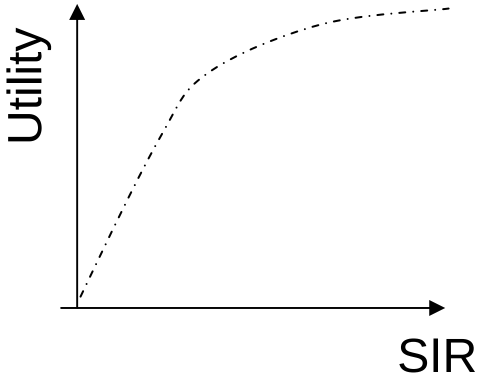
Game theoretic approaches to power control

- D. Famolari, N.B. Mandayam, D. Goodman and V. Shah, “A new framework for power control in wireless data networks: games, utility and pricing, Wireless Multimedia Network Technologies, Kluwer Academic Publishers, editors: Ganesh, Pahlavan, Zvonar, pp. 289-310, 1999.

- Wireless data QoS → different from voice
  - Different utility functions:



voice



data

Utility function

- General utility function: related to the energy per bit required for data transmissions
  - Assume coding → can detect all errors; can correct up to t errors
    → errors not corrected → packet retransmitted

Energy content of user j battery

Utility function:

SIR achieved by user j

$$U_j(P_j,\gamma_j)=\frac{E}{P_j}R_j\underbrace{f(\gamma_j)}$$

Measure of efficiency of
the transmission protocol

Rate at which information is transmitted

Specific utility function derivation

- SIR for CDMA:

System bandwidth

$$\gamma_j = \frac{W}{R}\frac{h_{jk}P_j}{\sum_{i \neq j} h_{ik}P_i + \sigma_k^2}$$

Transmission rate

- Information rate: $R_i = R\Gamma(t)$

code rate

- Efficiency function: may be frame success rate (FSR)?

$$FSR_j(\gamma_j, t) = \sum_{i=0}^{t} \binom{L}{i} \left(\frac{e^{-\frac{\gamma_j}{2}}}{2} \right)^{i} \left(1 - \boxed{\frac{e^{-\frac{\gamma_j}{2}}}{2}} \right)^{L-i}$$

Probability of error

L = number of encoded bits

Utility function – cont.

- FSR has practical meaning for utility
- Problem: user can obtain infinite utility for zero power
  - When the channel is extremely poor, the worst the receiver can do is to randomly guess the transmitted bit → Pe = 1/2
- Use an approximation for bit error rate function:

$$P_e = e^{-\frac{\gamma}{2}}$$

$$U_j(P_j,\gamma_j) = \frac{ER}{P_j}\underbrace{\frac{L-C-\log_2(L-C+1)}{L}}_{\text{code rate}}\sum_{i=0}^{t}\binom{L}{i}\left(e^{-\frac{\gamma_j}{2}}\right)^{i}\left(1-e^{-\frac{\gamma_j}{2}}\right)^{L-i}$$

User’s utility plot

- Level of interference – kept constant; utility plotted vs. transmitted power

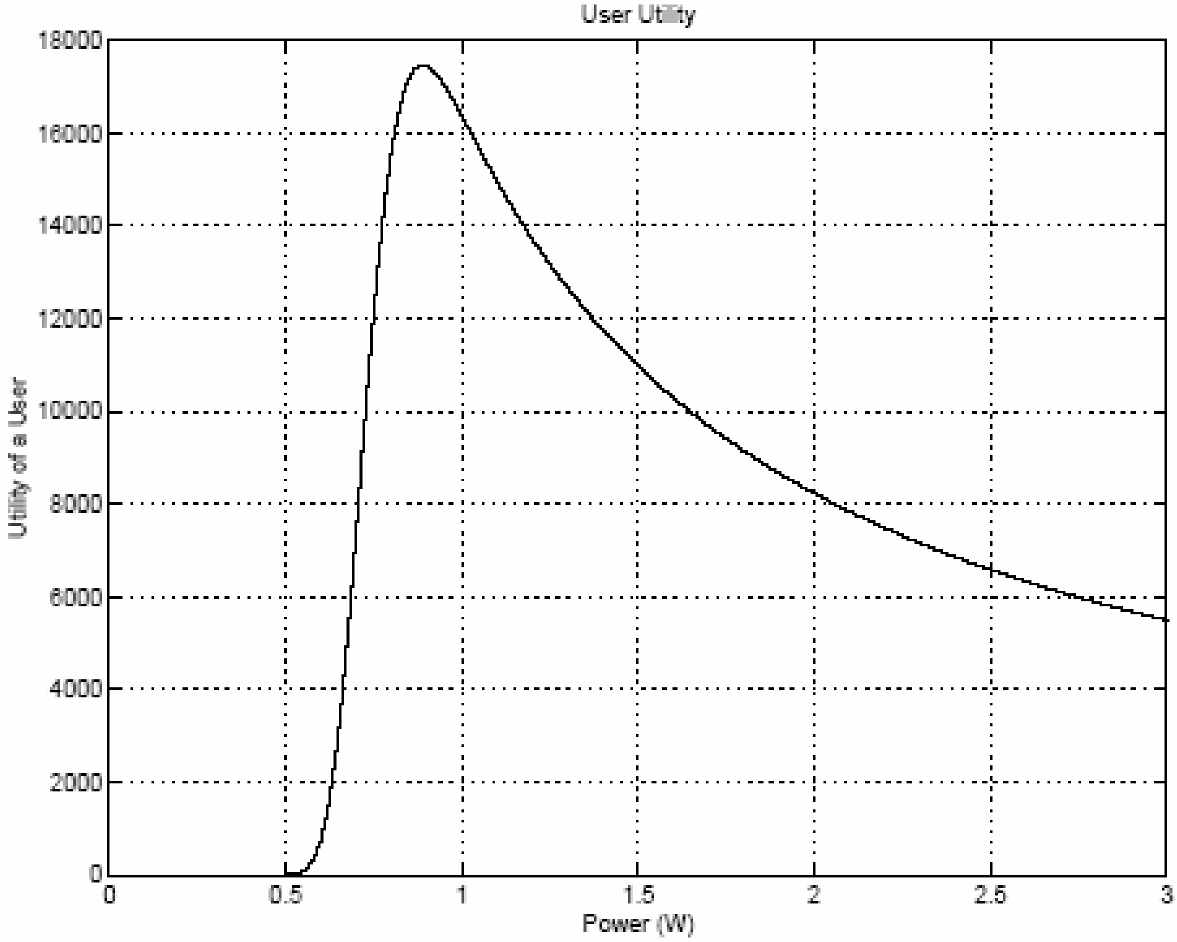


Figure 1.2 Utility function of a data user for fixed interference

Existence of Nash eq.

- The utility function – quasiconcave: Debreu’s theorem → there exists a pure strategy Nash equilibrium for the power control game
- Maximization of utility function:

$$f\left(\gamma^*\right) = \gamma^* f'(\gamma^*)$$

Optimal target SIR

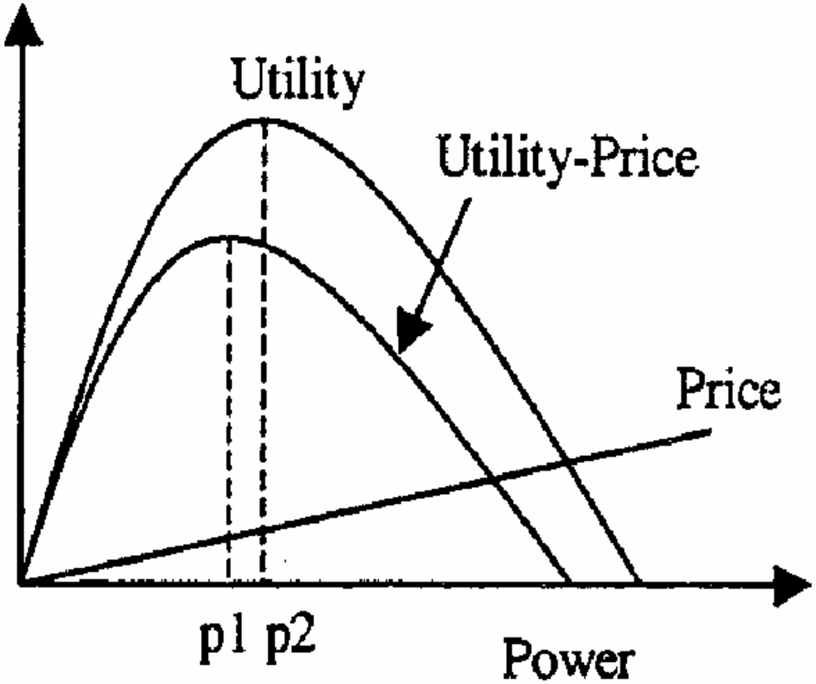
- The game theoretic solution converges to the same Nash equilibrium point as Yates’ framework for power control, when the target SIR is γ^*.
- Best response implementations
  - Using reaction functions

Pareto efficiency?

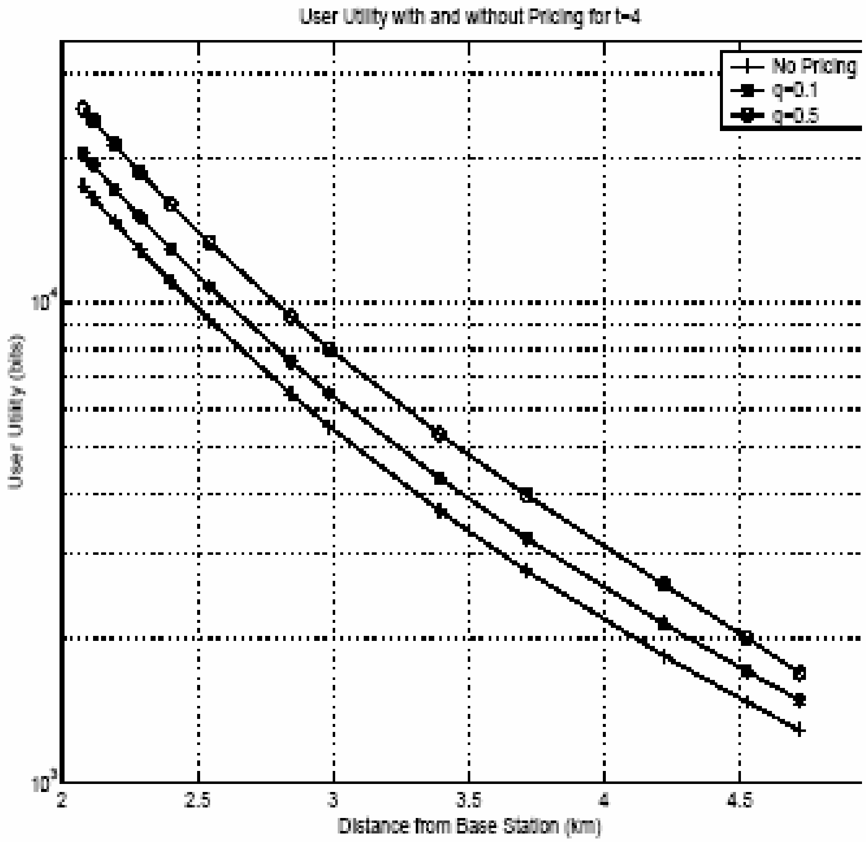
- Assuming that the noise factor is negligible compared with the interference → a scaling down of the powers for all users → higher efficiency
  - Still meet target SIR, but with lower energy consumption
- Mechanism: pricing → introduce linear pricing, proportional with transmission power

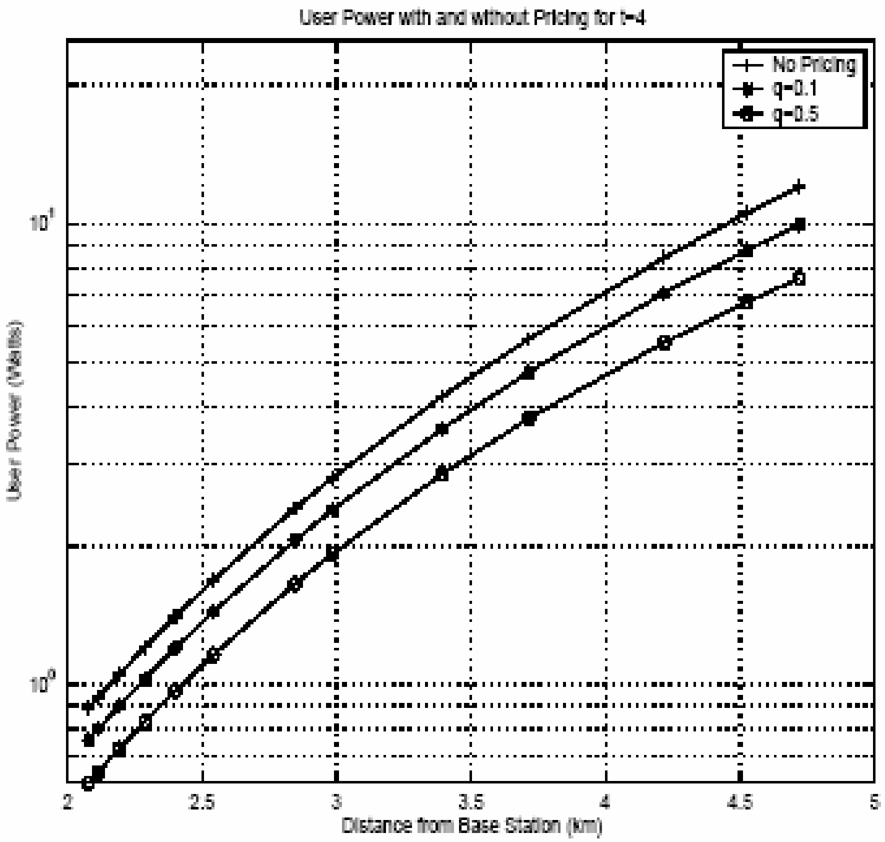
Price = qRP_j (t is in bits/Watt)

$$U_j^{'}\left(\gamma_j, P_j\right) = U_j\left(\gamma_j, P_j\right) - qRP_j$$



Some experimental results





Other approaches to power control

- A. Mackenzie, S. Wicker, “Game Theory in Communications: Motivation, Explanation and Application to Power Control
  - The Refereed Game
  - The Repeated Power Control Game

Based on a similar utility function derivation as before [Shah, Mandayam, Goodman], i.e., for an AWGN channel, non-coherent FSK modulation:

$$u_j(p_j,\gamma_j)=\frac{R}{p_j}\left(1-e^{-0.5\gamma_j}\right)^{L}$$

Number of bits in a data packet

The Refereed Game

- Base station “referees” the game by punishing users that are cheating
  - Desirable operating point is determined, under the assumption of equal received powers:

$$u_j(\tilde{p}_j) = \frac{ERh_j}{\tilde{p}_j}\left(1 - \exp\left(-\frac{W}{2R}\right)\frac{\tilde{p}_j}{(N-1)\tilde{p}_j + \sigma^2}\right)$$

  - $\tilde{p}_j =$ Pareto efficient received power
  - Base station punish users that use higher received power, by randomly inverting receiving bits for these users.

Punishment strategy

- If a user j's transmission is received with power $\tilde{p}_t + xW$, the SIR gain is

$$\Delta\gamma_j = \frac{W}{R}\frac{x}{(N-1)\tilde{p}_t + \sigma^2}$$

- The gain in BER: $\exp(-0.5\Delta\gamma_j)$

- Punishment: get same BER, but with a higher power → lower utility
  - Invert bits with probability

$$q_{bi} = \frac{\exp(0.5\Delta\gamma_j) - 1}{2(1 - \exp(-0.5\gamma_j))}\exp(-0.5\gamma_j)$$

Required Information exchange

- BS – inform users about target received power at each instant
- BS – feedback on the power levels each user is received

Alternate solution: the repeated game

- Provides incentives for cooperation
- All users are striving for the same operating point as the one derived for the referee game
- Strategy: Cooperate, unless one user cheats, then for one period revert to the power of the one-shot Nash eq. game, then return to cooperation
- Assumes infinite horizon game
- Discount factor close to 1
  - Packets in wireless networks come in quick succession, all packets are equally important

Comparisons: utility and transmission power

Fig. 4. Comparison of user transmit power.

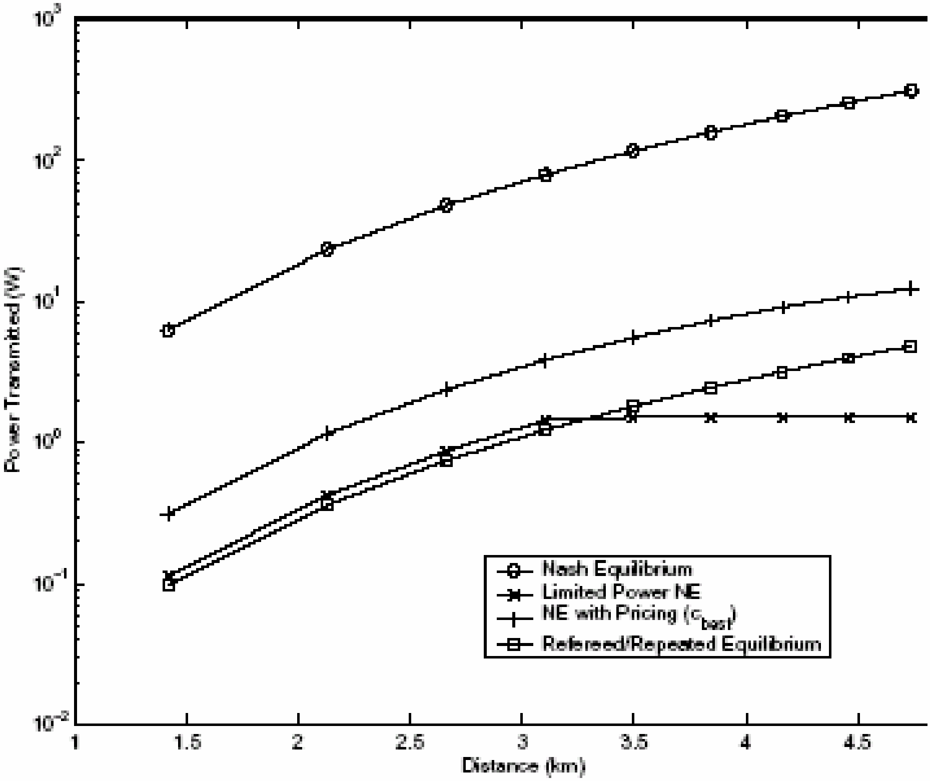
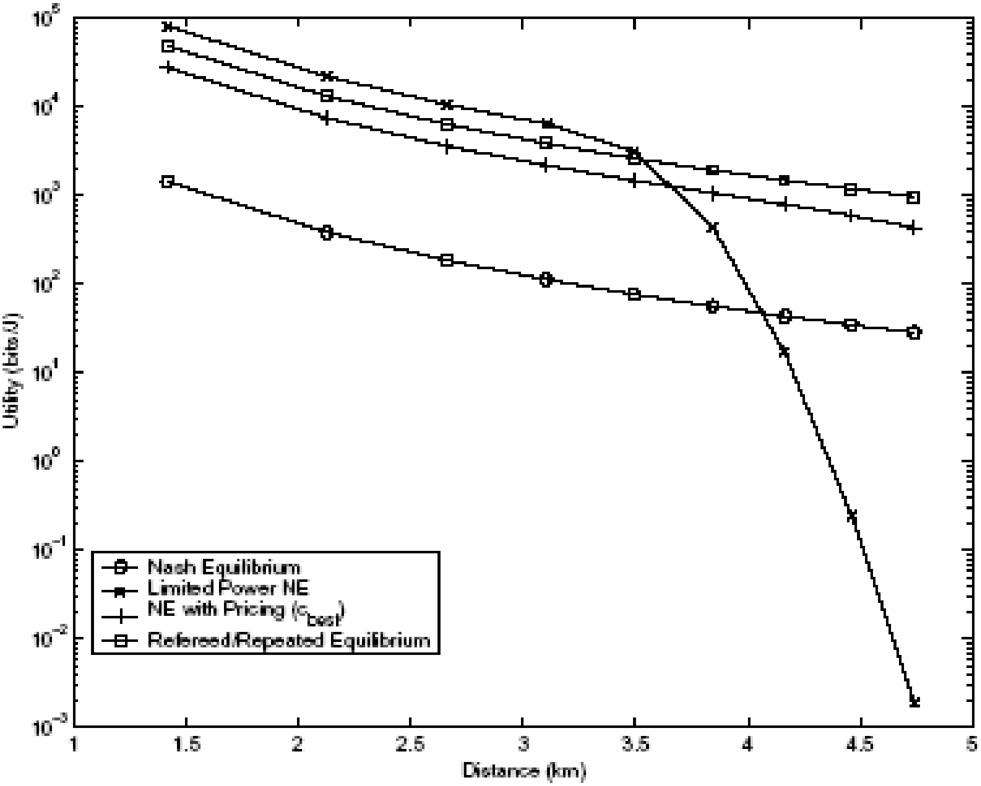


Fig. 3. Comparison of user utility.



Power control – potential game approach

- Achieved SIR: $$\frac{W}{R}\frac{h_{jk}P_j}{\sum_{i\neq j} h_{ik}P_i+\sigma_k^2}=SINR_{\text{target}}$$
- Consider the following utility function:

$$u_i(p_i,\sum_{j\in N\backslash i} p_j)=1-\left(SINR_{\text{target}}-h_i p_i+\frac{R}{W}\left(\sum_{j\in N\backslash i} h_j p_j\right)+\sigma^2\right)^2$$

- Verify that: $$\frac{\partial^2 u_i}{\partial p_i \partial p_j}=2\frac{R}{W}h_i h_j=\frac{\partial^2 u_j}{\partial p_j \partial p_i}$$
- Thus this game is a potential game

Power control – potential game approach

- Power control game converges to a pure strategy Nash equilibrium following a best response algorithm

- Nash equilibrium is given by maximizer of potential function:

$$f(\mathbf{p}) = 2\sum_{k\in N}(SINR_k h_k p_k - h_k^2 p_k^2) + 2\frac{R}{W}\sum_{k\in N}\sum_{m=k+1}^{n} h_k h_m p_k p_m$$

- Reference: http://www.mprg.org/people/gametheory/Meetings.shtml,

“Distributed power control and game theory”

Observations

- Utility function choice for power control is not unique
  - It depends on QoS specifications for users and network
  - To ensure good properties (convergence, unicity, etc), utility function needs to satisfy some mathematical properties (shown for various cases in the first part of the class)

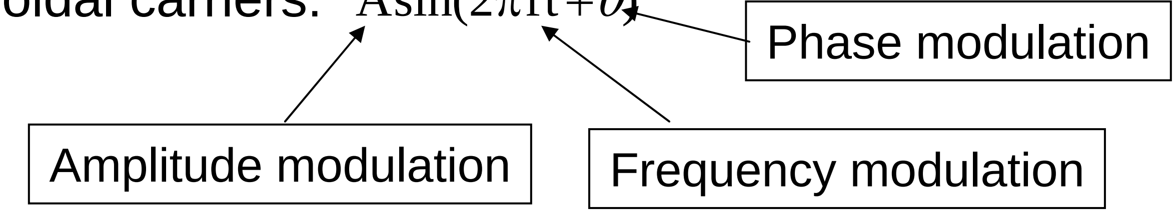
  - Some more examples:
    - [Basar, 2001]: utility function is defined as a difference between a linear pricing and a term that is proportional with Shannon capacity
    - [Chong 2001, Xiao 2003]: a sigmoid model for throughput is factored in the utility definition

References

- [Shah, Mandayam, Goodman]: V. Shah, N.B. Mandayam, D.J.Goodman, “Power control for wireless data based on utility and pricing”, Proceedings of the IEEE International Symposium on Personal, Indoor and Mobile Radio Communications, pp. 1427-1432, September 1998.
- [Basar, 2001]: T. Alpcan, T. Basar, R. Srikant, E Altman}, “CDMA uplink power control as a non-cooperative game”, Proceedings of the 40th IEEE Conference on Decision and Control, pp. 197-202, December 2001
- [Chong 2001]: M. Xiao, N. Shroff, E. Chong}, “Utility-Based Power Control in Cellular Wireless Systems”, Proceedings IEEE INFOCOM'2001, pp. 412-421
- [Xiao 2003]: M. Xiao, N. Shroff, E. Chong, “ A Utility-Based Power Control scheme in Wireless Cellular Systems”, IEEE/ICM Transactions on Networking, April 2003, vol 11, no 2, pp. 210-221.

Adaptive modulation

- Recall – we transmit symbols over the air by modulating a carrier
  - Sinusoidal carriers: $A\sin(2\pi ft+\theta)$



  - Quadrature modulation: we use as carriers both sin and cos
  - M-ary modulation: $A_1, A_2 = \pm 1, \pm 3, \ldots, \pm\sqrt{M} - 1$
    - Example M - QAM
      - $\log_2 M$ bits encoded into one symbol
  - Large M – higher rates!!!
    - Higher BER too → need higher SNR/SIR → better quality channel

Adaptive modulation

- BER target → application specific: imposed

- Channel quality fluctuates → change modulation to meet target BER
  - Fading
  - Interference changes (burstiness, mobility, access control, etc.)

- Channel quality fluctuates → change transmission power to achieve target BER

- Adaptive modulation and power control: related → joint optimization??

Joint link adaptation and power control

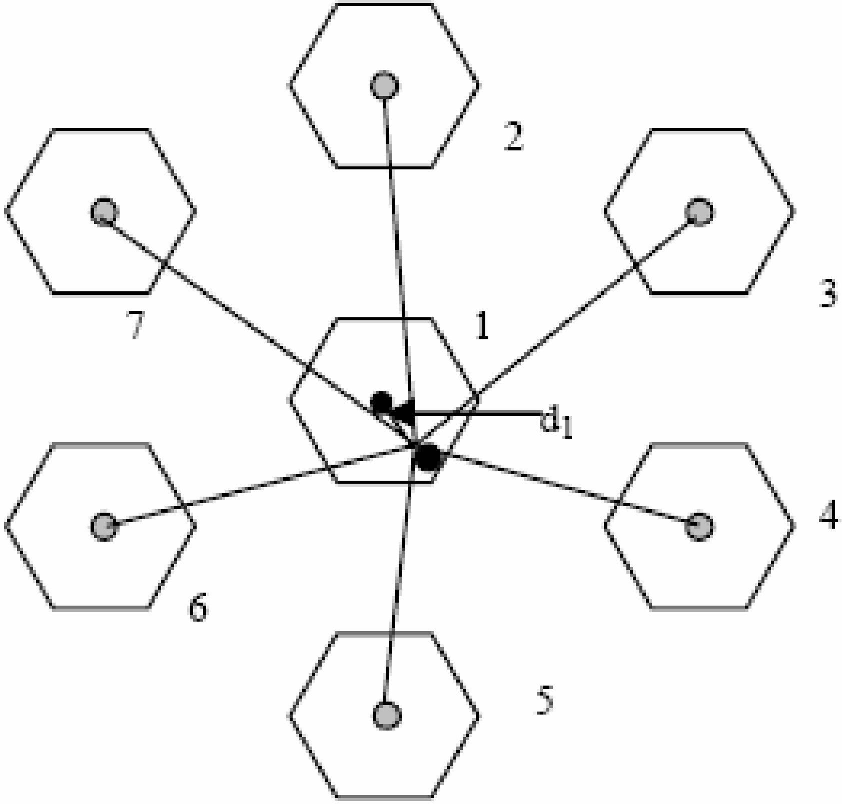
- Reference: “Game theoretic Analysis of Joint Link Adaptation and Distributed Power Control in GPRS”, S. Ginde, J. Neel, R. M. Buehrer, VTC, fall 2003.

- Application for GPRS
  - General Packet Radio Service → standard for wireless data communications → extension from GSM (Global System for Mobile Communications): digital cellular system, based on TDMA technology

System model – cellular with TDMA technology

Similar as in our SIR analysis:
- First tier of interferers
- cellular reuse factor: 3

-Compute SIR in dB for user i:



SIR

$$\gamma_i = 10\log_{10}\left(\frac{G_{ii}P_i}{\sum_{j\neq i}^{N} G_{ij}P_j + \eta_i}\right)$$

Transmitted Power for user j

link gain (path loss)

noise at receiver i

Performance measure

- Performance measure: analytical model for throughput
  - Depends on the SNR(SIR)
  - Can be modeled by a sigmoid function of the SIR, with some parameters: A, λ, and δ → fitted for given modulation and coding

$$L(\gamma) = \frac{A}{1 + \exp[-\lambda(\gamma - \delta)]}$$

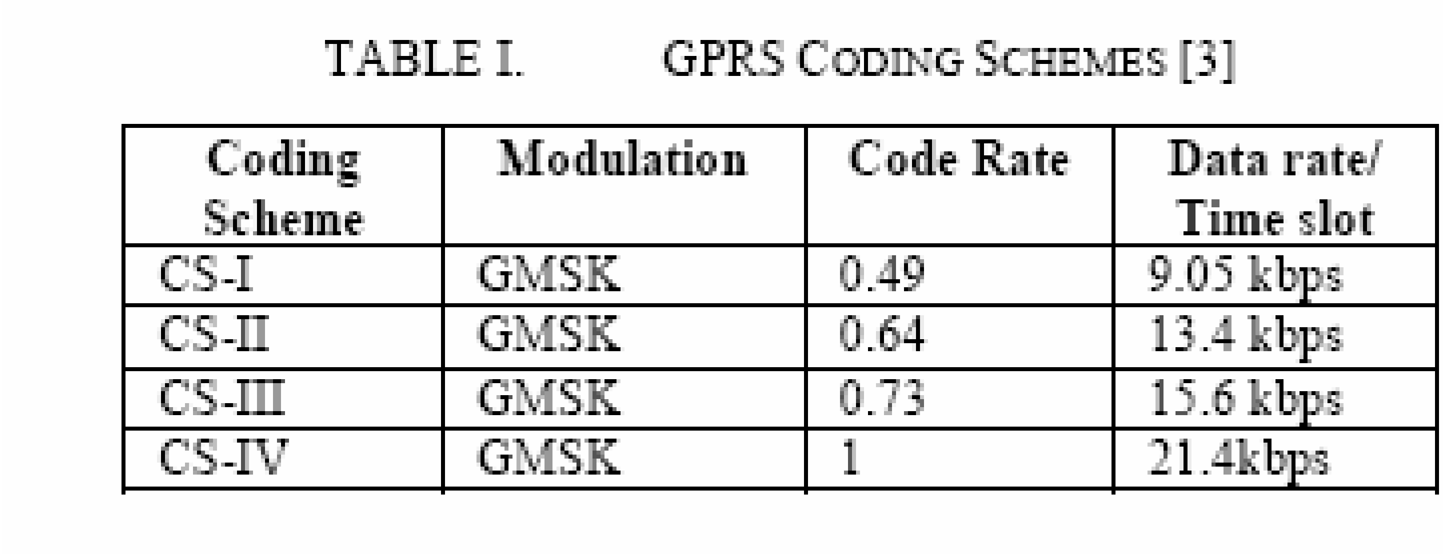
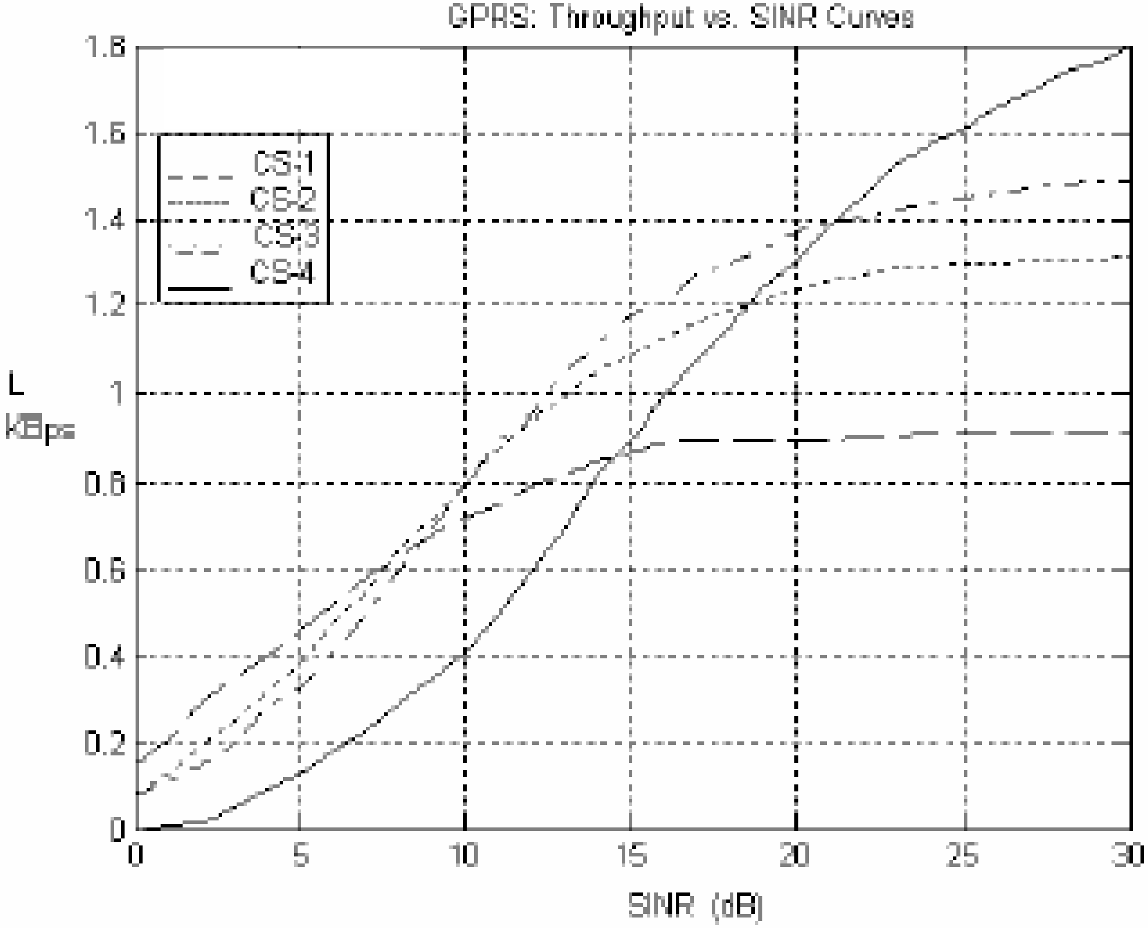
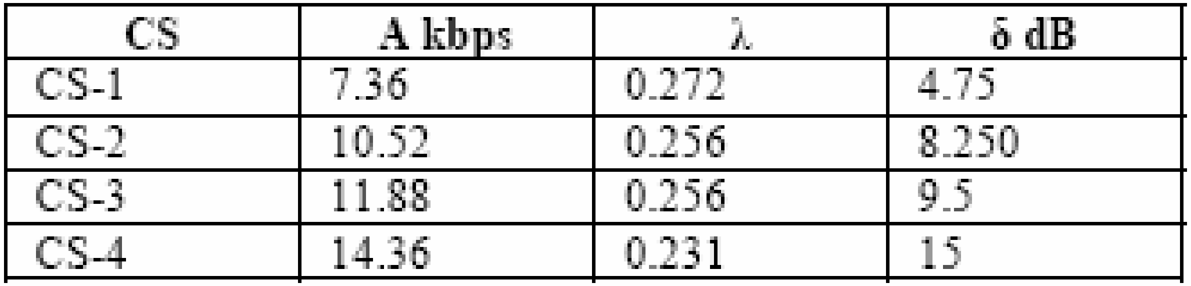


TABLE I. GPRS CODING SCHEMES [3]

Coding Scheme	Modulation	Code Rate	Data rate/ Time slot
CS-I	GMSK	0.49	9.05 kbps
CS-II	GMSK	0.64	13.4 kbps
CS-III	GMSK	0.73	15.6 kbps
CS-IV	GMSK	1	21.4kbps

Performance measure: Throughput





CS	A kbps	λ	δ dB
CS-1	7.36	0.272	4.75
CS-2	10.52	0.256	8.250
CS-3	11.88	0.256	9.5
CS-4	14.36	0.231	15

Game theoretic formulation

- Players: set of co-channel links
- Actions: select powers and rates:

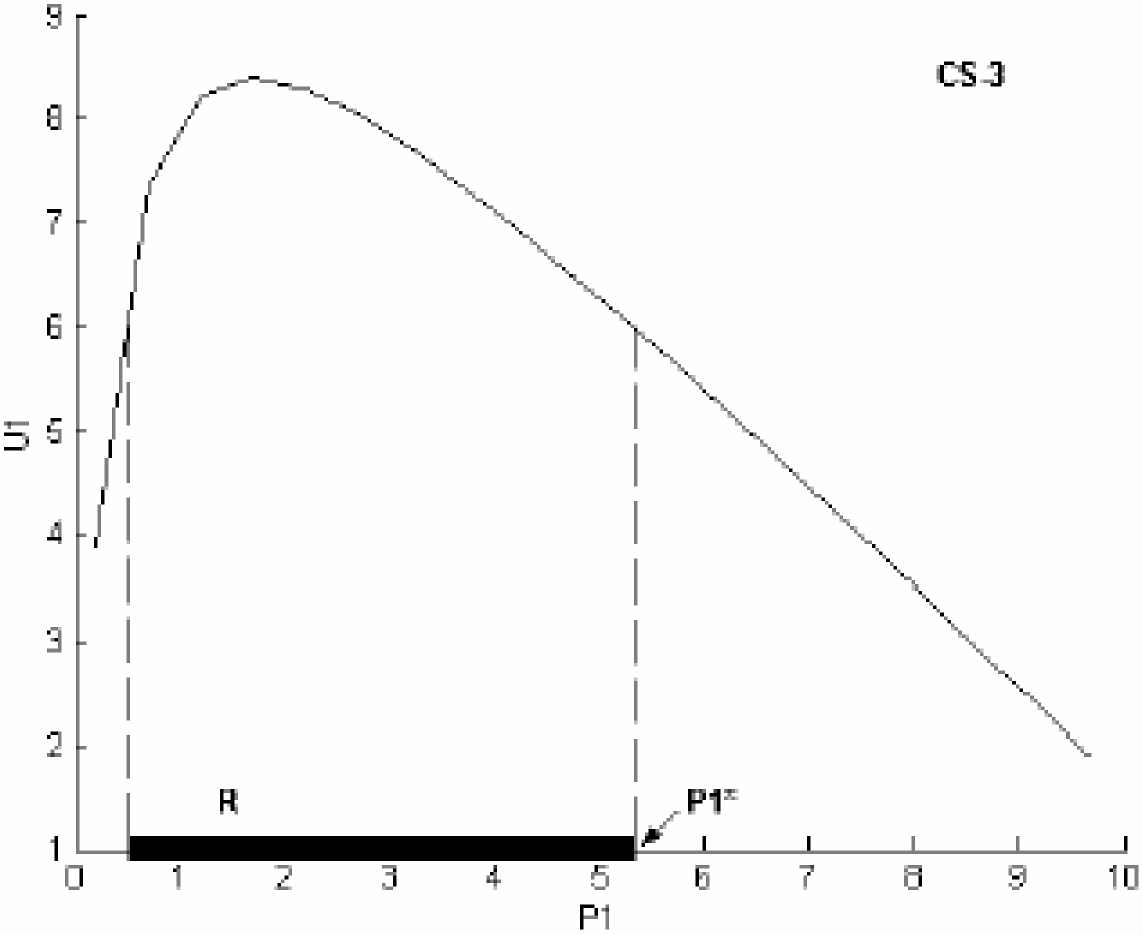
$$A_i = \{(P_i, r_i) \mid P_i \in \lfloor P_{i,\min}, P_{i,\max} \rfloor\}, r_i \in \{r_1, r_2, r_3, r_4\}$$

- Utility function: throughput measure with pricing function

$$U_i(\mathbf{P}, r_i) = \frac{A(r_i)}{1 + \exp\{-\lambda(r_i)[\gamma_i - \delta(r_i)]\}} - KP_i^q$$

Existence of Nash equilibrium

- If keep rates constant (select only powers) → Debreu's theorem holds → guaranteed to have at least 1 Nash equilibrium



Joint power control and rate assignment solution

- Two step optimization:
  - start with some initial values for powers and rates
  - Step1: select power to maximize utility function

$$P_{i,eqm} = \max_{P_i} U_i(P_i, P_{-i}, r_i)$$

  - Step 2: for the given selected powers, determine rates

$$r_{i,eqm} = \arg\max_{r_i} U_i\left(r_i, \mathbf{P}_{eqm}\right)$$

  - Continue until no improvement in rates can be achieved

Figures of merit definition

- FOM1: ratio between system throughput and sum of the fractions of peak power consumed by links

$$FOM1 = \frac{\sum_i L_i}{\sum_i \frac{P_i}{P_{\max}}}$$

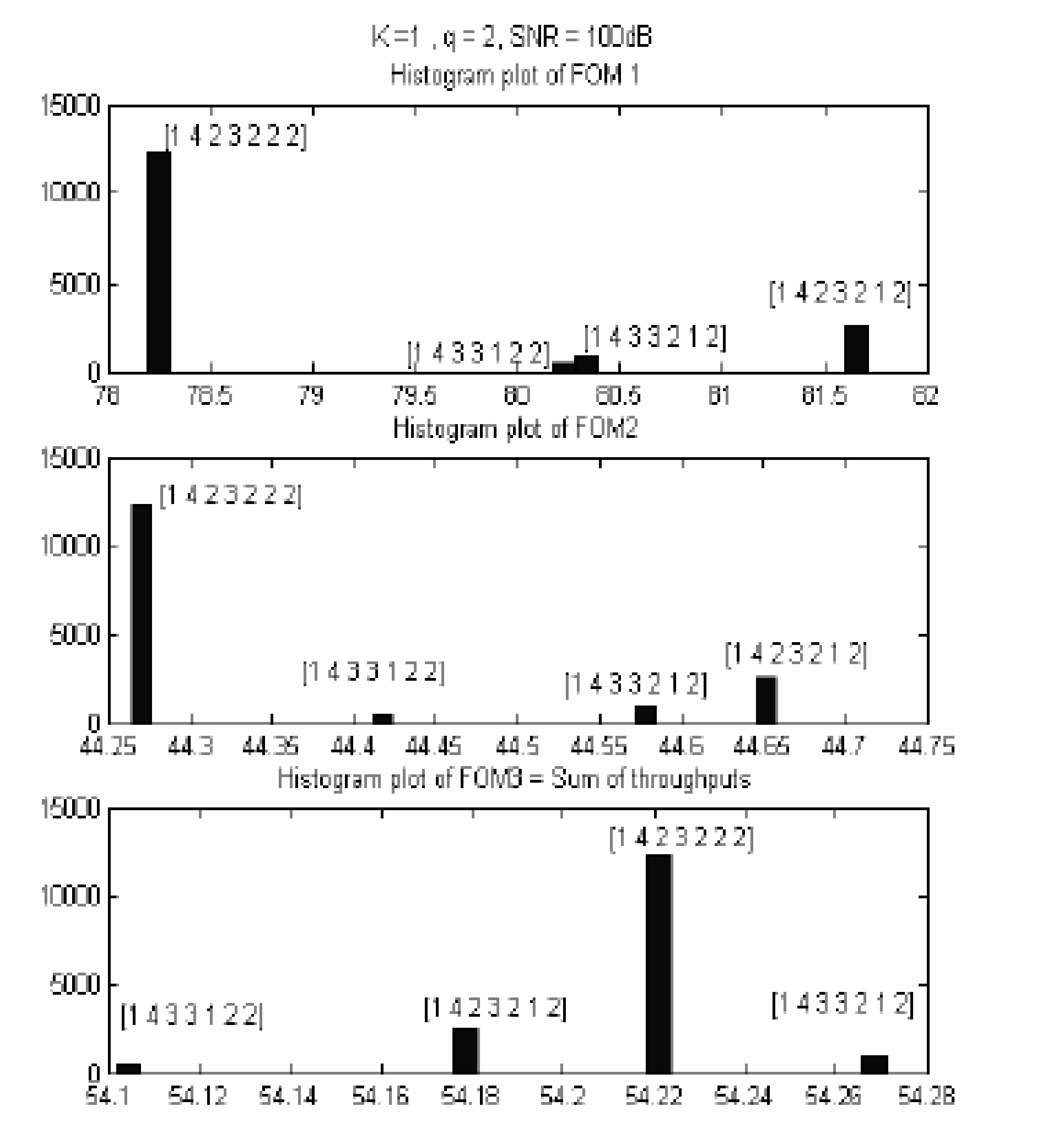
- FOM2: difference between system throughput and sum of the fractions of peak power consumed by links, scaled by the peak throughput

$$FOM2 = \sum_i L_i - A_{\max} \sum_i \frac{P_i}{P_{\max}}$$

- FOM3: system throughput

$$FOM3 = \sum_i L_i$$

Nash equilibrium: not unique



Some performance results

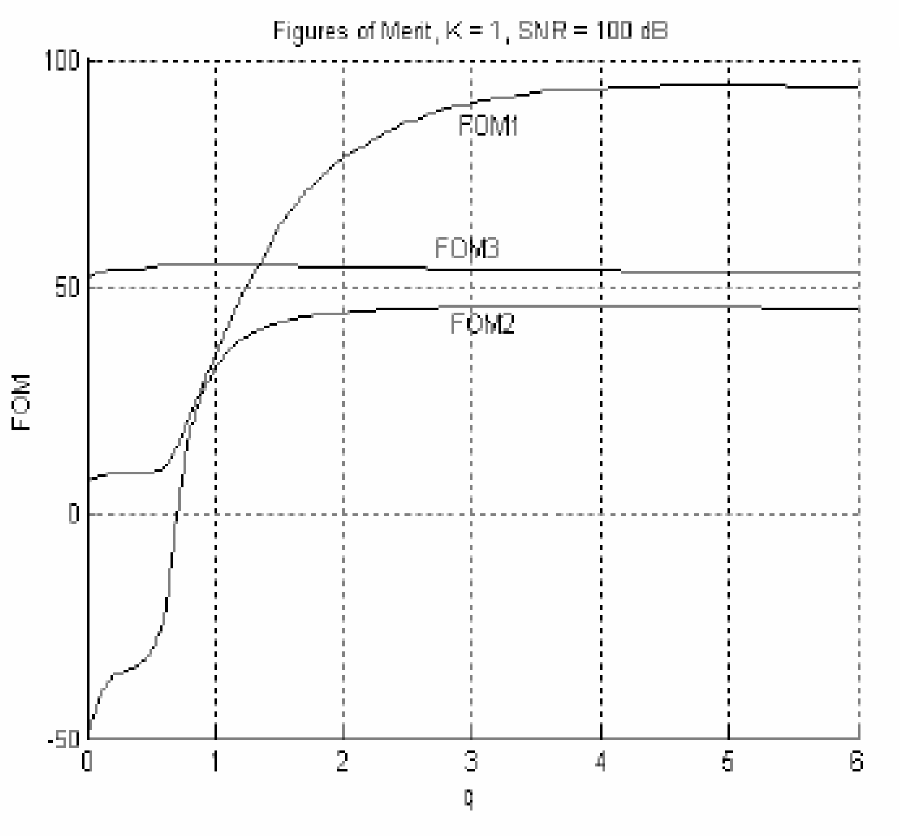


Figure 5. Effect of *q* on FOM's, SNR = 100 dB

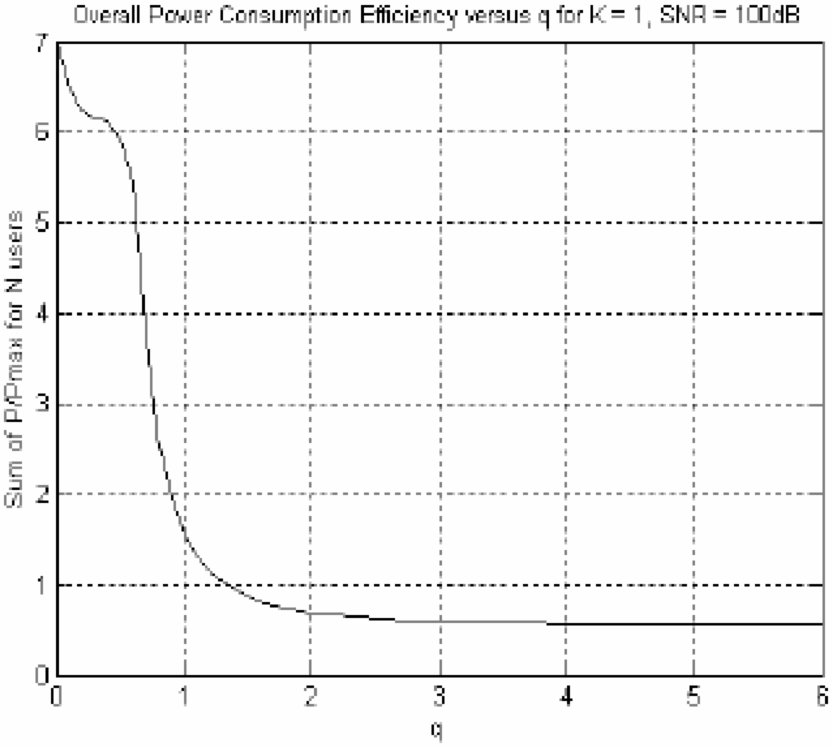


Figure 6. Effect on *q* on power consumption

More performance results

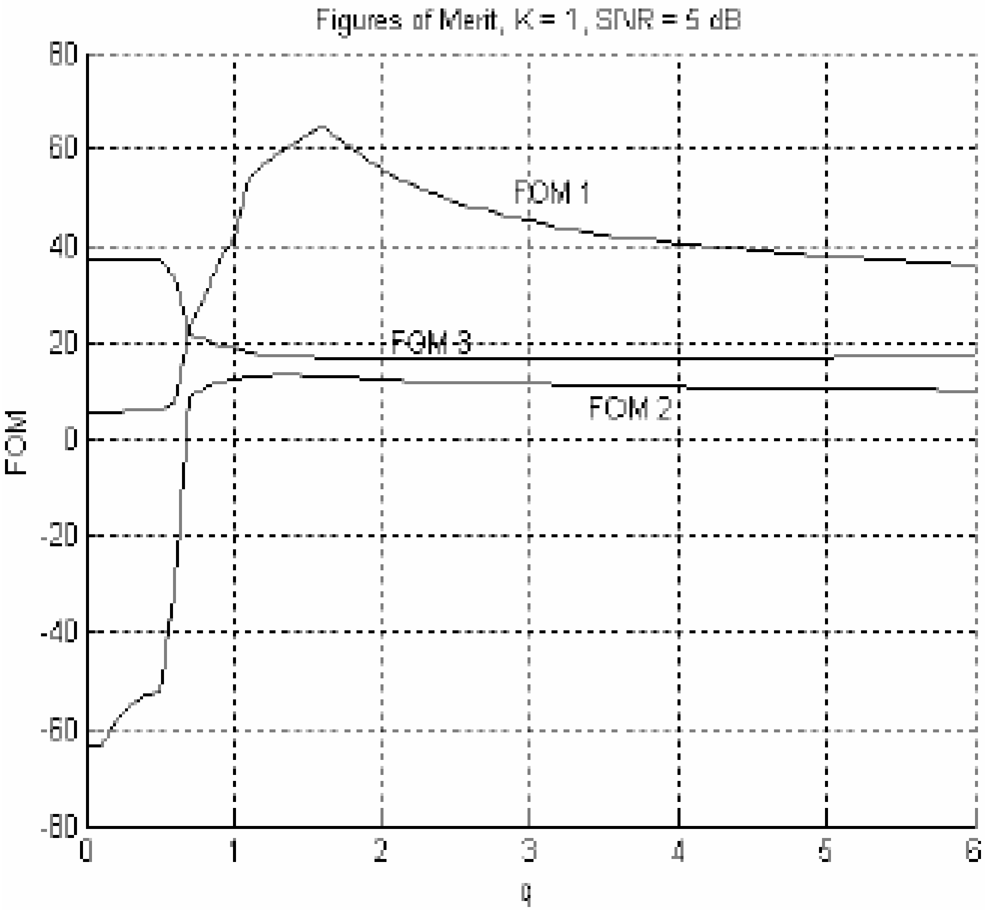


Figure 7. Effect of *q* on FOM's, SNR = 5 dB

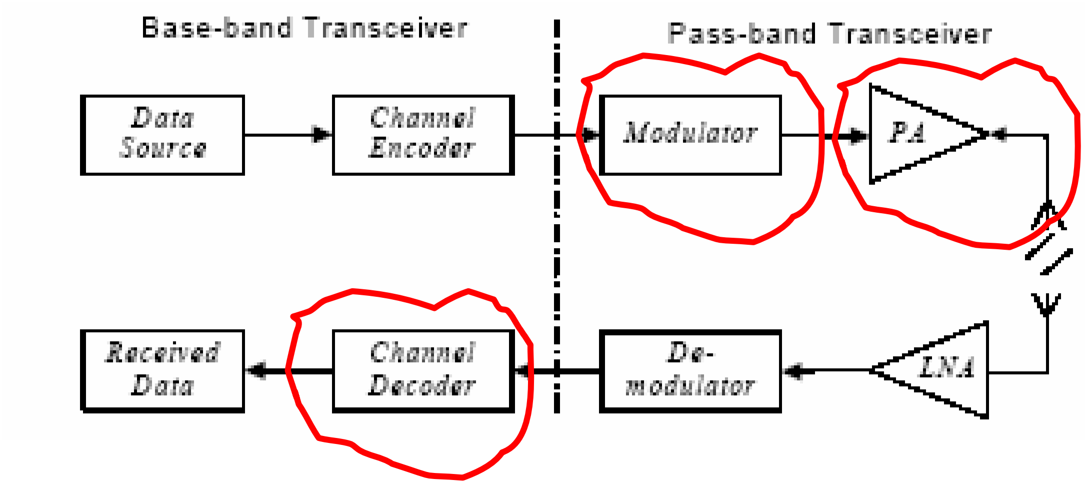
Dynamic energy minimization in wireless transceivers

- Reference: “A Game Theoretic Approach to Dynamic Energy Minimization in Wireless Transceivers”, A. Iranli, H. Fatemi, M. Pedram, Int’l Conference on Computer Aided Design, November 2003, pp. 504-509.

- System model: ad hoc network, OFDM (Orthogonal Frequency Division Multiplexing) transmission (802.11a)
  - OFDM = multicarrier transmission technique, which allows a more tight packing of multiple carrier frequencies, by keeping the carriers orthogonal
  - A signal is split into sub-signals that are transmitted in parallel on the multiple carriers

Optimization problem

- Minimize total energy consumption for a link
  - (A) Minimize energy at transmitter
  - (B) Minimize energy at receiver



Optimization problem

- No interference is considered
- Transmitter: can select powers and modulation levels → influences the total energy expenditure
- Receiver: can choose decoder’s parameters:
  - For convolutional codes → Adaptive Viterbi decoder → accuracy of decoder increases as the number of decoding stages increases; however, the power consumption of the decoder increases also
  - Parameters chosen: Truncation length (number of paths consider to find the optimum path) of the decoder
- Receiver and transmitter alone cannot optimize energy independently, depend one of another
- Optimization modeled as a Stackelberg game → multi-level optimization problem formulated

Optimization problem

- Strategy for the receiver: $x \in X$

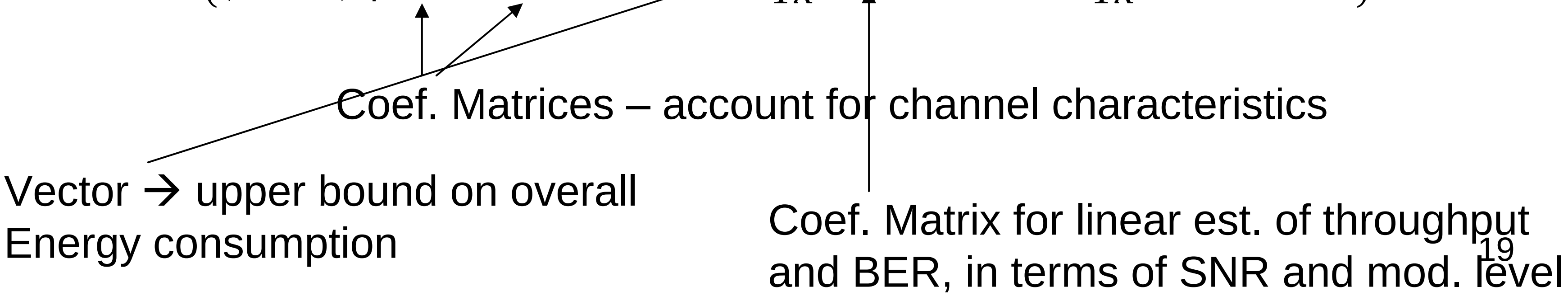
$$X = \{(TL_1, TL_2, ...TL_n \mid \forall i : TL_i \in TLS\}$$

- Strategy for transmitter: $y \in Y$

$$Y = \{P_{Tx}, b_1, b_2, ..., b_n \mid i : b_i \in MLS, P_{Tx} \in PLS\}$$

- Optimization:

$$\{(\hat{X}, \hat{Y}) \mid A\hat{X} + B\hat{Y} < R\hat{E}Q_{Tx}, D\hat{Y} \geq R\hat{E}Q_{Tx}, \hat{X}, \hat{Y} \geq 0\}$$



Coef. Matrices – account for channel characteristics

Vector → upper bound on overall
Energy consumption

Coef. Matrix for linear est. of throughput
and BER, in terms of SNR and mod. level

Some performance results

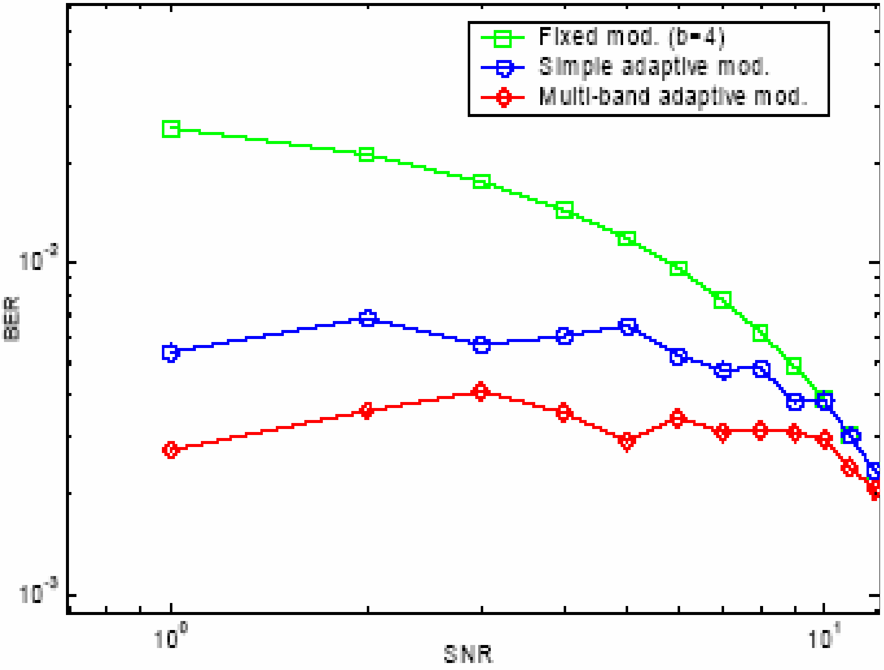
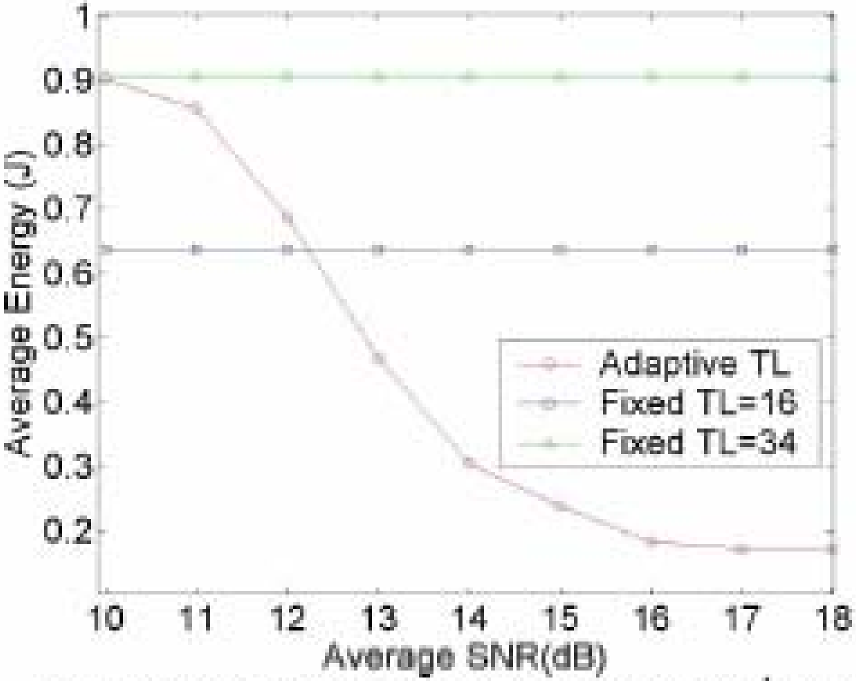


Figure 5. Average BER as a function of SNR for adaptive and fixed modulations



(a) Average Energy vs. SNR (BER=10^{-4} & b=4

More performance results

$$E_{avg} = \alpha \cdot E_{Transmit} + (1 - \alpha) \cdot E_{Receive}$$

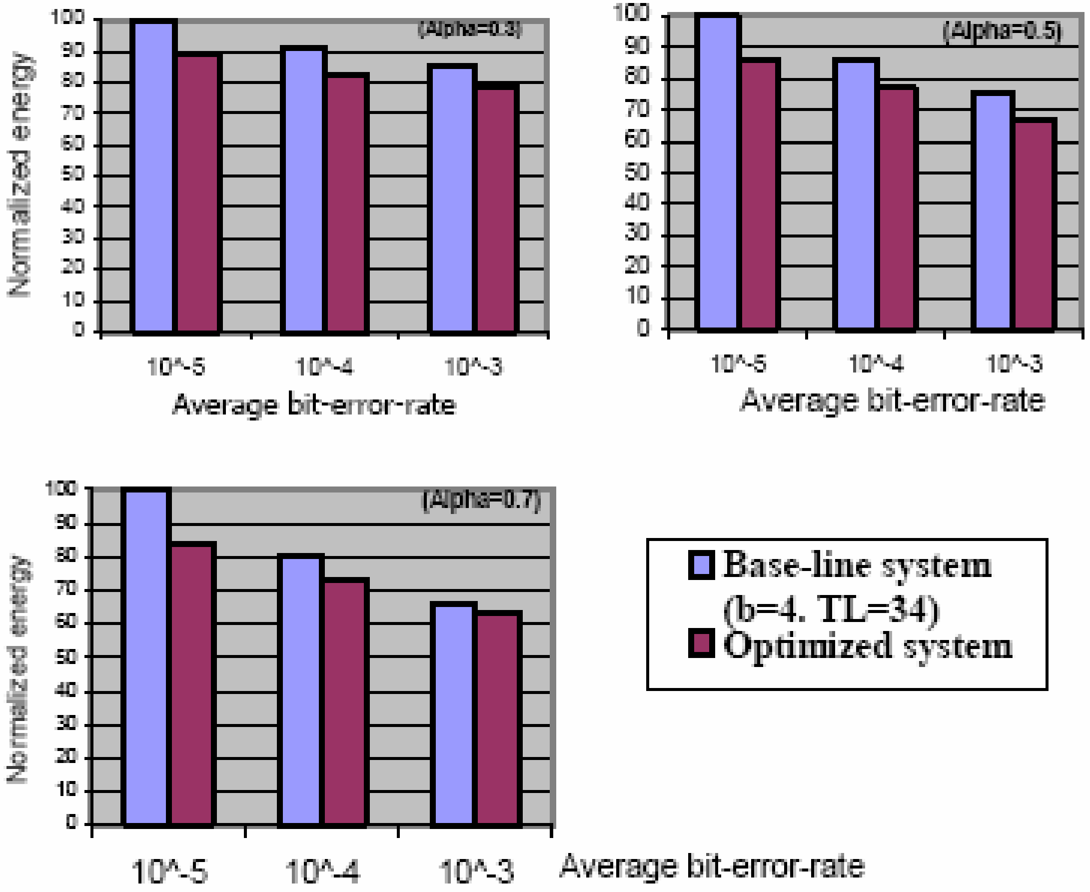


Figure 7. Normalized energy consumptions

Network assisted power control

Reference:

- **David J. Goodman and Narayan B. Mandayam**, "Network Assisted Power Control for Wireless Data", Mobile Networks & Applications, vol. 6, No. 5, pp. 409-415, 2001

Network assisted power control for wireless data

- Non-cooperative power control with pricing:
  - Linear pricing → effective policing mechanism that influences user behavior towards a more efficient operating point
  - Unfair equilibrium?
    - Users settle to unequal SIRs → users with better channel conditions obtain higher utilities, higher SIRs, and lower transmit powers
  - Can a feedback mechanism from a centralized controller improve performance/fairness?

Game theoretic model

- Utility function
  - (L = information bits, M = packet size, including coding bits, f(γ) ≅ probability of a successful transmission)

$$U = \frac{RL}{M}\frac{f(\gamma)}{P} \text{ bits/Joules}$$

- N terminals – share same physical channel
  - Single cell CDMA → uplink
- SIR:

$$\gamma_i = \frac{W}{R}\frac{P_i h_i}{\sum_{\substack{j=1\\ j\neq i}}^{N} P_j h_j + \sigma^2} = G\frac{P_i h_i}{\sum_{\substack{j=1\\ j\neq i}}^{N} P_j h_j + \sigma^2}$$

spreading gain

- Nash eq. obtained for γ* which maximizes utility fct.

NAPC (Network Assisted Power Control)

- All terminals adjust their powers → common target γ_T
- What is γ_T ?
- CDMA → interference limited systems → upper bound on the number of terminals that can simultaneously operate at target SIR

$$N(\gamma_T) \leq 1 + G/\gamma_T \Leftrightarrow \gamma_T \leq G/(N-1) = B$$

bandwidth expansion

- Fairness condition: equal received SIR → equal received powers

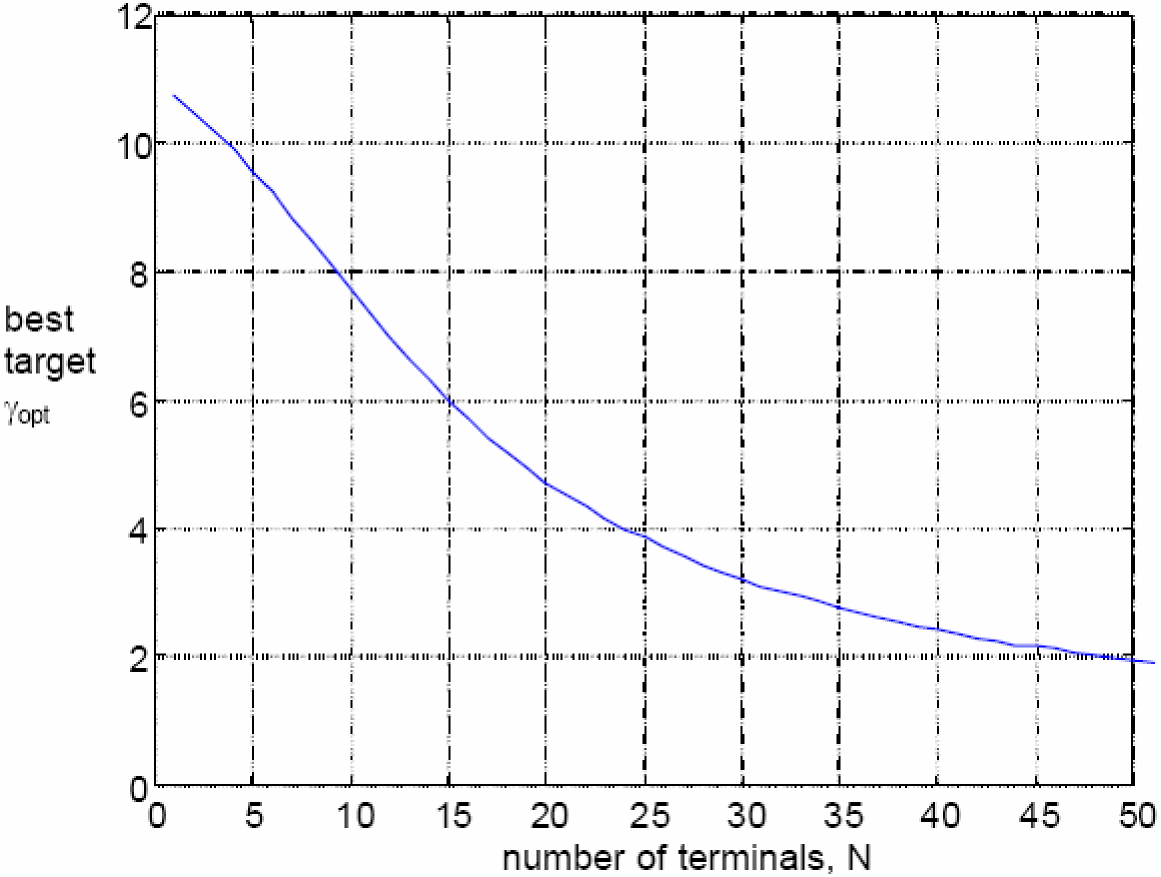
$$\gamma_T = \frac{GP_{rec}}{(N-1)P_{rec} + \sigma^2} \Rightarrow P_{rec} = \frac{\gamma_T \sigma^2}{G - (N-1)\gamma_T} = P_i h_i$$

New utility function

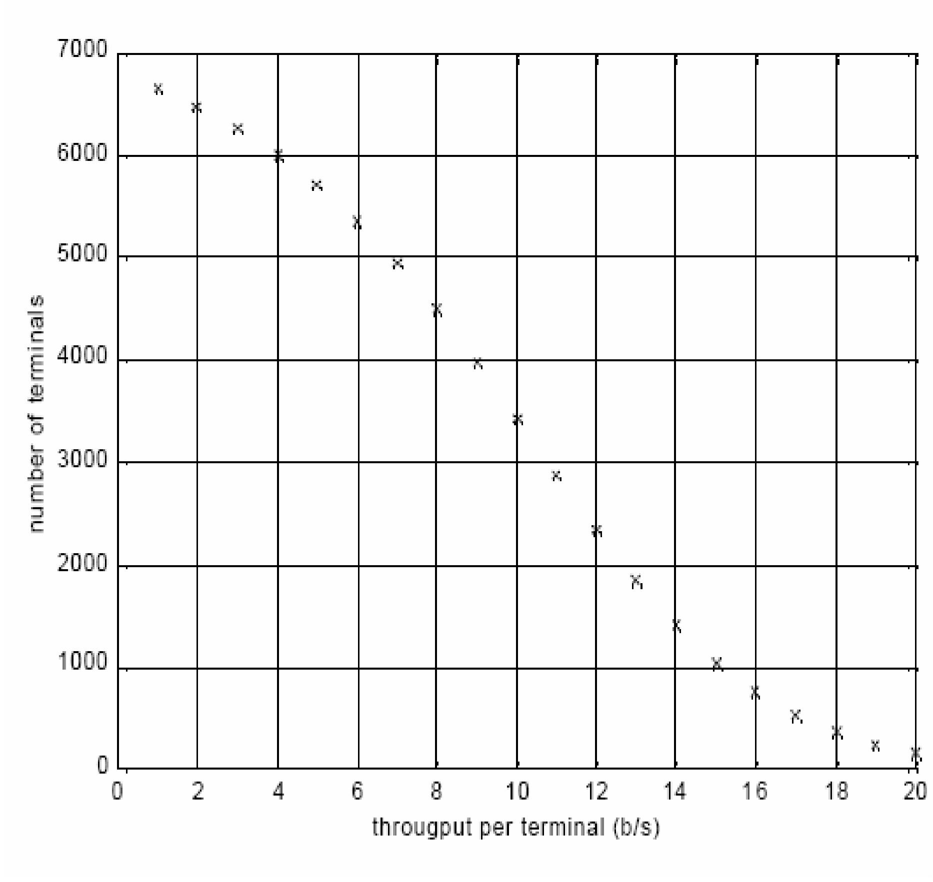
$$U_i = \frac{LR}{M} = \frac{f(\gamma_T)}{\dfrac{\gamma_T \sigma^2}{h_i\left[G-(N-1)\gamma_T\right]}} = \frac{LR}{M}\frac{h_i}{\sigma^2}\, f(\gamma_T)\left[\frac{G}{\gamma_T-(N-1)}\right]$$

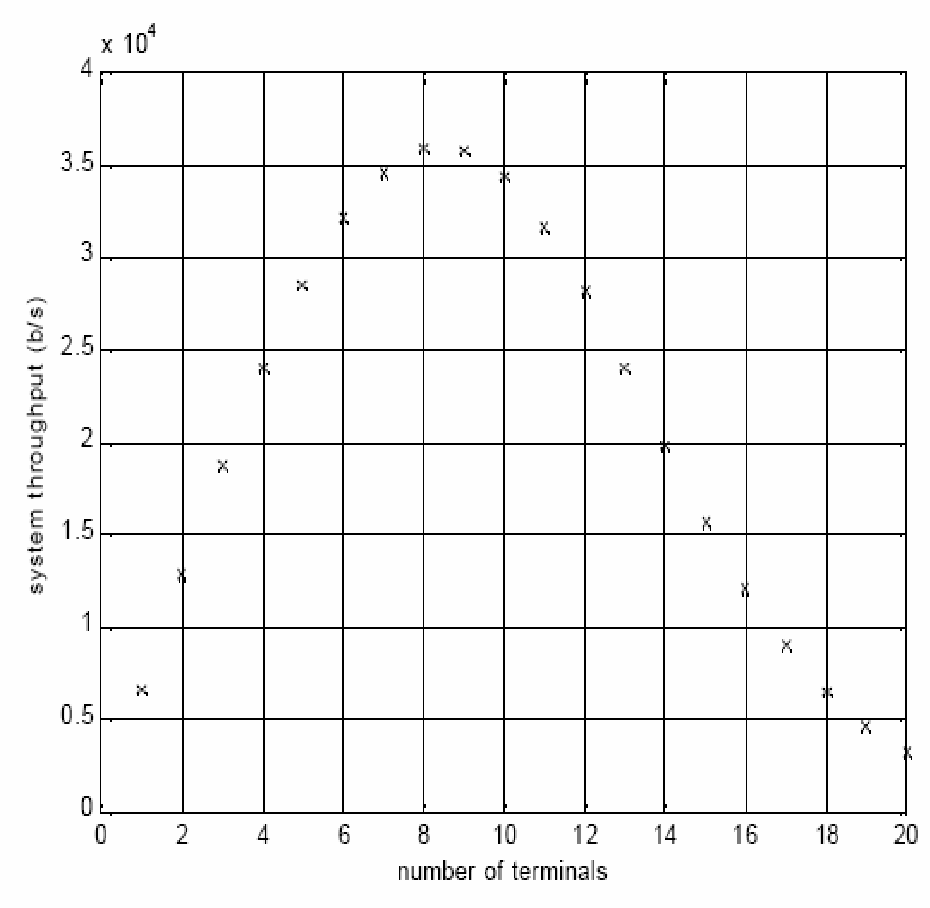
- Observations:
  - All users get maximum utility for same common value of the target SIR → terminals need to change powers s.t. achieve γ_{opt}
    - γ_{opt} depends on the number of terminals in the system → need feedback information from a centralized controller
  - Users' utilities depend on their relative position to the base station (through their path gains)

Some simulation results

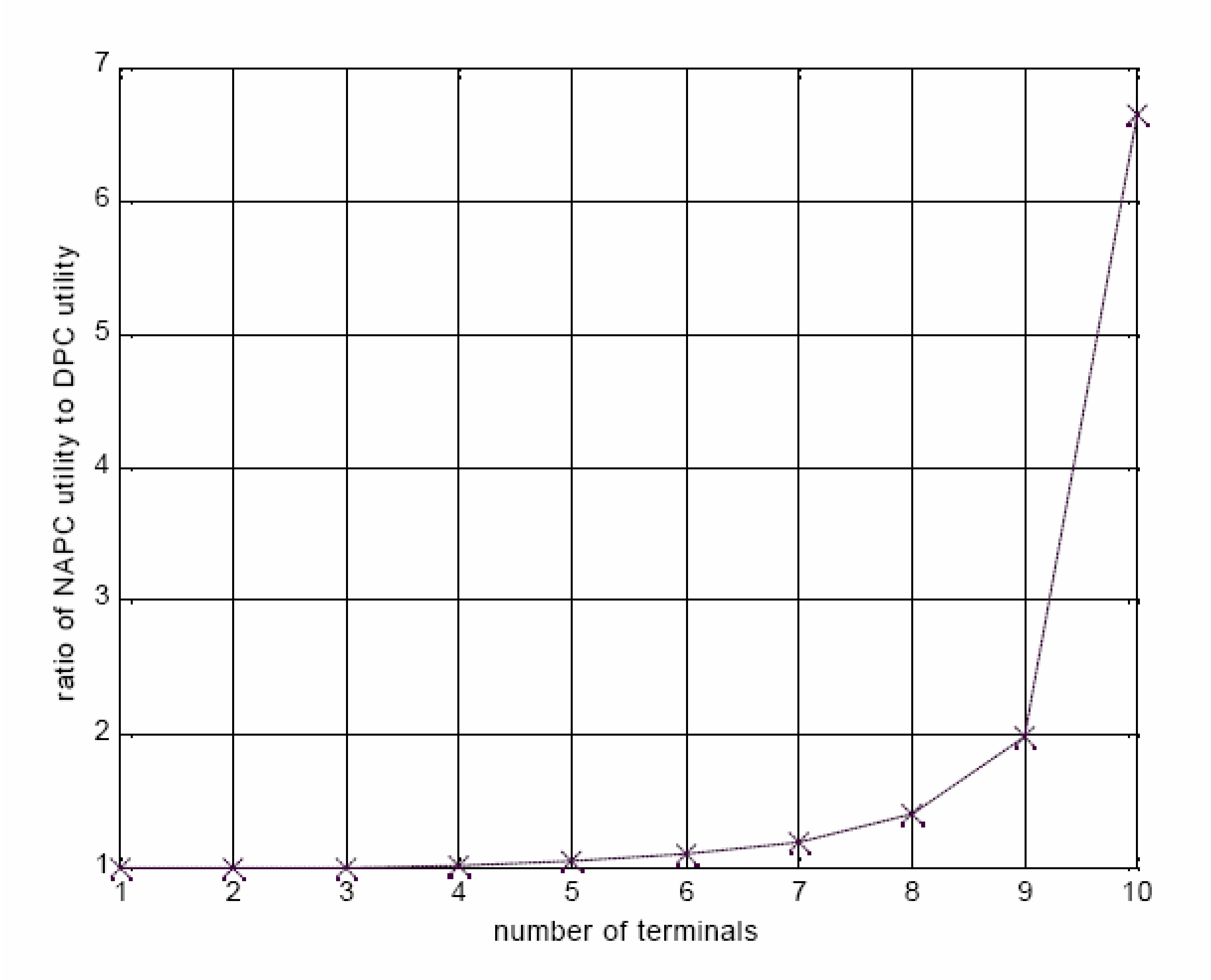


Throughput


7000
6000
5000
4000
3000
2000
1000
0
number of terminals
0
2
4
6
8
10
12
14
16
18
20
througput per terminal (b/s)


x 10^4
4
3.5
3
2.5
2
1.5
1
0.5
0
system throughput (b/s)
0
2
4
6
8
10
12
14
16
18
20
number of terminals

Comparison with distributed power control algorithm



Admission/Access control

- The performance of wireless systems, depends on how many users share the channels
- Idea: limit the number of users that can simultaneously use the shared channel
  - Different time scales
    - At connection time level: admission control → admit user only if all users (including the new one) can meet QoS specifications (e.g. SIR, delay, average delay, throughput, etc) for the life of the connection
    - At packet level – access control (MAC) → schedule users (deterministically or randomly) to transmit, such that the interference is limited to desired levels.

Call admission control – game theoretic approach

Reference:

- J. Hou, J. Yang and S. Papavassilliou, “Integration of pricing with call admission control to meet QoS requirements in cellular networks”. IEEE Trans. Parallel and Distributed Systems, Vol. 13, No. 9, Sep. 2002.

CAC – system model

- Calls arriving in the systems
  - New calls → perf. measure: new call blocking probability
    - Poisson arrivals, λ_n
  - Handoff calls → perf. measure: handoff call blocking probability
    - Poisson arrivals, λ_h
- Duration of calls: exponentially distributed
- Assume guard channel scheme: reserve channels for handoff
- Average number of admitted users, $N = f(\lambda_n)$
  - $0 \leq f(\lambda_n) \leq C$, where C is the total numbers of channels
  - $f'(\lambda_n)>0, f(\lambda_n=0)=0, \lim_{\lambda n \rightarrow \infty} f(\lambda_n)=C$
  - Differentiable, monotonically increasing, continuous in λ_n
- Probability of blocking
  - $P_b=\alpha P_{nb}+\beta P_{hb}=g(\lambda_n)$
  - $\alpha + \beta = 1$, $\beta>\alpha$ reflects higher cost of blocking for handoff
  - $0 \leq g(\lambda_n) \leq 1$, $g'(\lambda_n)>0, g(\lambda_n=0)=0, \lim_{\lambda n \rightarrow \infty} g(\lambda_n)=1$

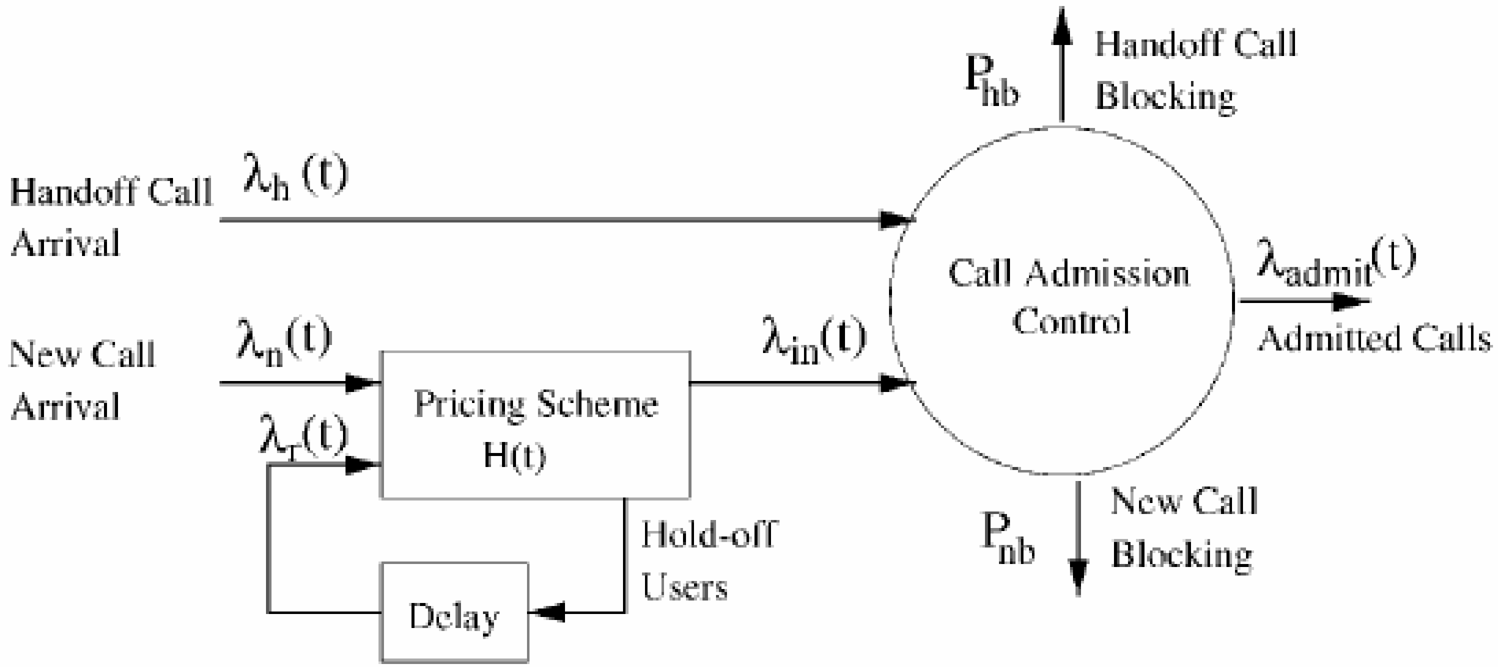
Utility

- User’s utility, $U_s = h(P_b)$
  - Differentiable, monotonically decreasing, concave in P_b
  - $h(P_b) \geq 0, h'(P_b) < 0, h''(P_b) < 0$
  - $U_s(P_b = 0) = U_s^{max}$

- Utility is maximized for λ_n^* :
  - $U = N\ U_s = f(\lambda_n) * h(P_b) = f(\lambda_n) * h(g(\lambda_n))$
  - based on assumptions on *f, g* and *h*

- $\lambda_n > \lambda_n^*$ → users blocked, resources overused
- $\lambda_n < \lambda_n^*$ → resources wasted

System model



Idea: introduce dynamic pricing (function of congestion) to regulate offered traffic.

H(t) → percentage of incoming users who accept price at time *t:*

$$(\lambda_n + \lambda_r)H(t) = \lambda_{in}(t) \quad (*)$$

Pricing policy

- If light load **($\lambda_n < \lambda_n^*$)** → charge nominal price
- If high load **($\lambda_n \geq \lambda_n^*$)** → charge dynamic price depending on demand

Demand function *D(p(t)):* $$D(p(t)) = e^{-(\frac{p(t)}{p_0} - 1)^2}$$

p(t) = price charged at time *t*
p0 = nominal (normal) price
D(p) = percentage of users that accept price *p*
D(p0) = 1 (normal price accepted by 100% of users)

Determine price

- From (*): $H(t) = \frac{\lambda_{in}}{\lambda_n(t) + \lambda_r(t)} \leq \frac{\lambda_n^*}{\lambda_n(t) + \lambda_r(t)}$
- Demand function: $D(p(t)) = e^{-(\frac{p(t)}{p_0} - 1)^2}$
- *H(t)=D(p(t))* → $p(t) = D^{-1}\left(\min\left(\frac{\lambda_n^*}{\lambda_n(t) + \lambda_r(t)}, 1 \right) \right)$

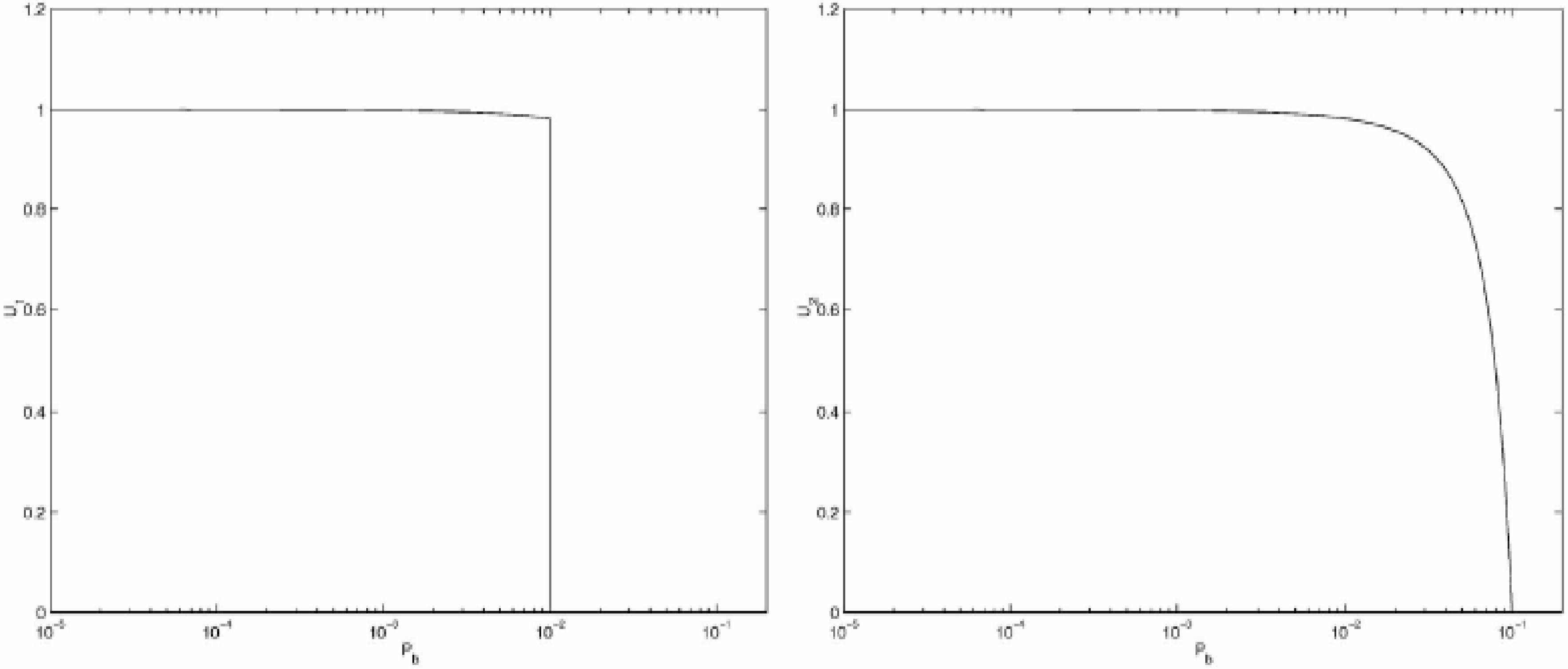
Simulation results

- Two different utility functions proposed
- Hard QoS specifications $$U_1 = \begin{cases} 1 - e^{30(P_b - 0.1)} & \text{when } 0 \le P_b \le 0.01 \\ 0 & \text{when } P_b > 0.01 \end{cases}$$
- Soft QoS specifications $$U_2 = \max(1 - e^{30(P_b - 0.1)}, 0).$$
- Different users behaviors:
  - PSwHR → users that do not accept current price may choose to retry; blocked users are cleared
  - PSwR → all holdoff and blocked users may choose to retry
  - PSwRL →part of the holdoff and blocked may leave the system and the others may choose to retry

Utility functions

U1

U2

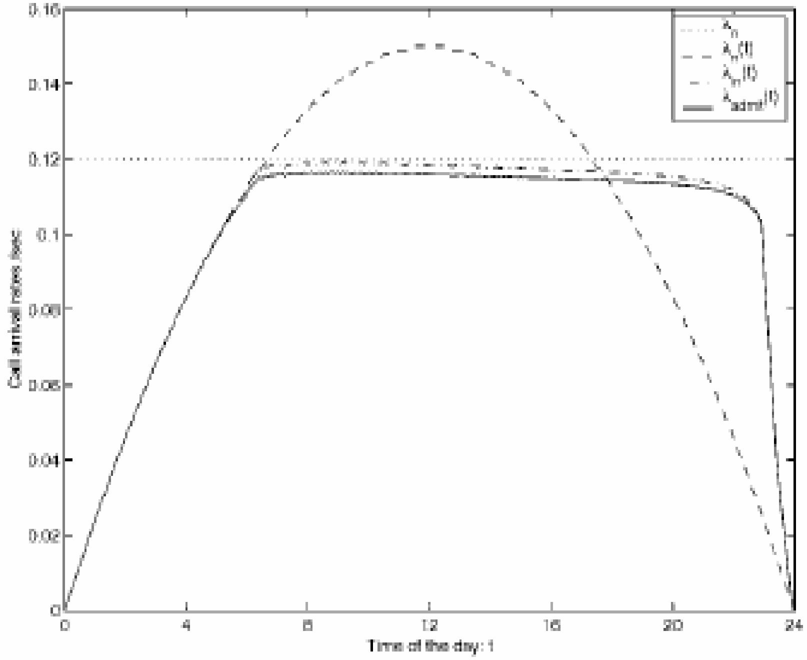


(a)

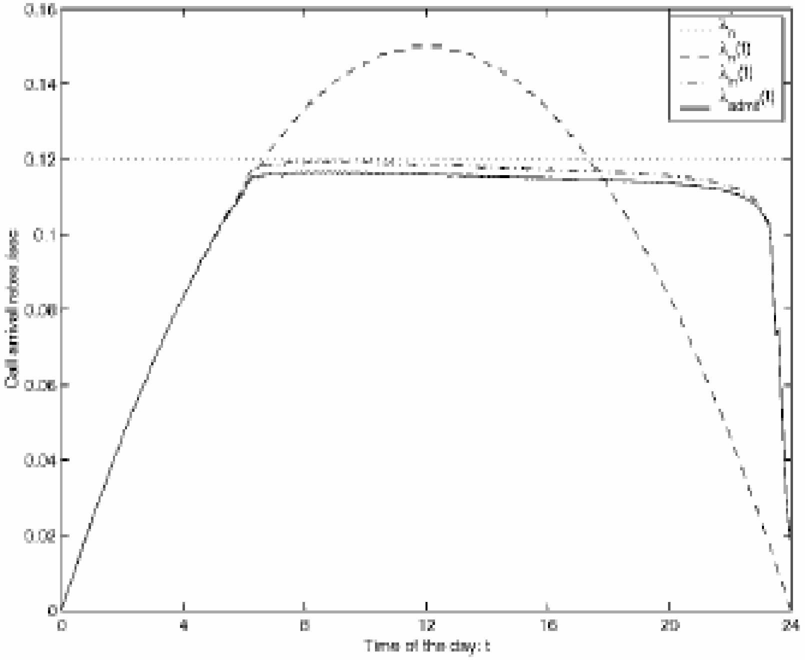
(b)

Offered load

PSwHL

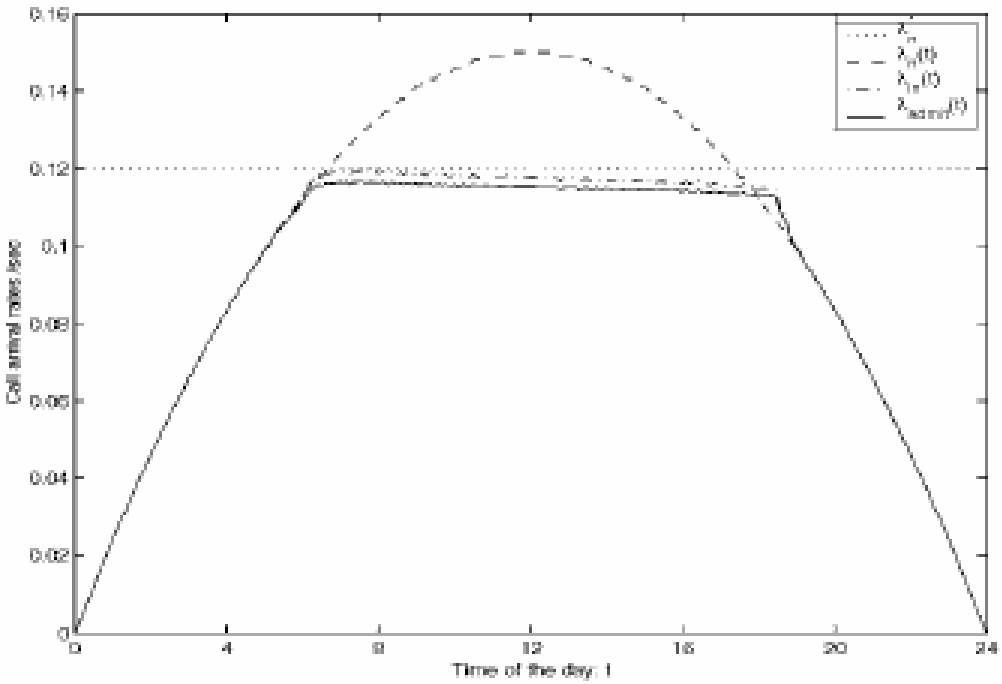


PSwR



Admitted load

PSwRL

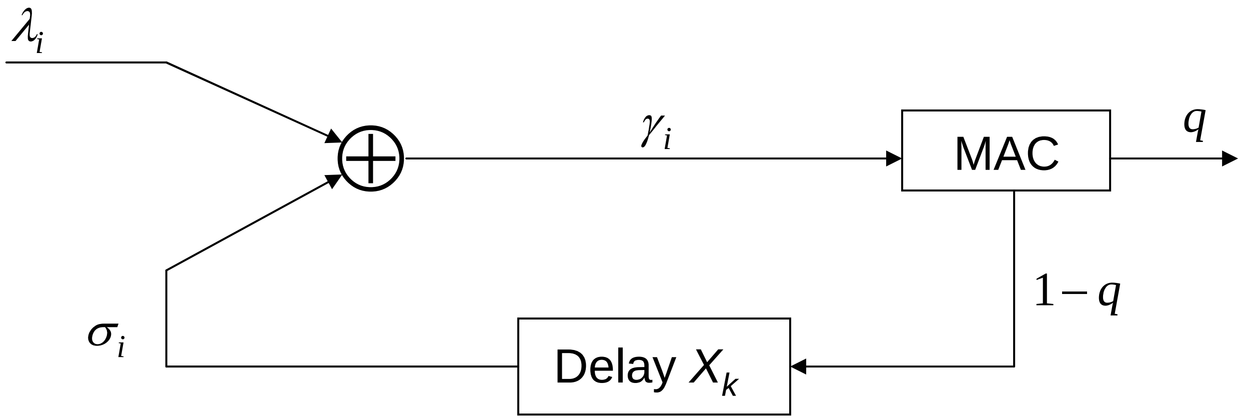


MAC: Slotted Aloha

- First and simplest packet access protocol
- First developed for collision channels (if two packets are sent in the same time they collide and are lost)
- Analysis was later on extended to account for other channel models
- Simple protocol
  - If there is a message, transmit it
  - If successful, remove message from the queue
  - If collision, wait a random time and retry
- Slotted Aloha
  - confine transmission at the beginning of well defined slots with duration equal to the message transmission time.
  - Transmission occurs with probability *p* at the beginning of each time slot

Slotted Aloha – approx. performance analysis

- Protocol model at one particular node



  - Arrivals: Poisson with mean λ_i
  - Approx: σ_i Poisson, more accurate if $E[X_k] = E[X] = \xi$ large
  - q – probability of successful transmission
  - For equilibrium:

$$q\gamma_i \leq \lambda_i$$

Slotted Aloha analysis: cont.

- *p* = probability of transmitting

$$q = (1-p)^{M-1} \approx (1-p)^{M}$$

- *p* = Pr{at least one arrival in the slot} = 1-Pr{no arrival in the slot}:

$$p = 1 - e^{-\gamma_i} \Rightarrow q = e^{-M\gamma_i} = e^{-\gamma}$$

$$\lambda = \sum_{i=1}^{M} \lambda_i = M\gamma_i e^{-\gamma} = \gamma e^{-\gamma} \quad (*)$$

$$\lambda \leq \max \gamma e^{-\gamma} = e^{-1} \Rightarrow \lambda^* = e^{-1}$$

Max throughput

- Message delay
  - N = number of the retransmission attempts: geometric r.v. (iid retransm)

$$E[D] = \sum_{n=0}^{\infty} E[D \mid N = n] P[N = n] = \sum_{n=0}^{\infty} (1 + nE[X]) P[N = n] =$$

$$= \sum_{n=0}^{\infty} P[N = n] + E[X] \sum_{n=0}^{\infty} nP[N = n] = 1 + \xi E[N]$$

Slotted Aloha analysis: cont.

- Average number of retransmissions and average delay:

$$E[N] = \frac{1}{q} - 1 \qquad E[D] = 1 + \xi\left(e^{\gamma} - 1\right)$$

- From $(*)$ solve for λ -> 2 solutions, 2 states of equilibrium:
  - Low channel activity – almost all transmissions successful
  - large number of backlogged packets, frequent retransmissions, low probability of success

– It can be shown that as ξ increases, the system stabilizes: the probability of ending up in the high delay state decreases; however, large ξ -> large average delay as well.