POTENTIAL GAMES: A FRAMEWORK FOR VECTOR POWER CONTROL PROBLEMS WITH COUPLED CONSTRAINTS

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ABSTRACT

In this paper we propose a unified framework, based on the emergent *potential games* to deal with a variety of network resource allocation problems. We generalize the existing results on potential games to the cases where there exists coupling among the (possibly vector) strategies of all players. We derive sufficient conditions for the existence and uniqueness of the Nash Equilibrium, and provide different distributed algorithms along their convergence properties. Using this new framework, we then show that many power control problems (standard and *non-standard*) with coupled constraints among the users, can be naturally formulated as potential games and, hence, efficiently solved. Finally, we point out an interesting interplay existing between potential games, classical optimization theory, and Lyapunov stability theory.

1. INTRODUCTION

Power control in flat-fading CDMA (or TDMA/FDMA) systems (cellular or ad-hoc), where each user has only one variable (degree of freedom) to optimize (i.e. the transmit power), is by now well understood. In fact, it can be either elegantly recast in the framework of the so called "standard" problems (in the sense defined in [1]) or equivalently interpreted as a strategic non cooperative game [2, 3, 4]. In both cases, sufficient conditions for the existence and uniqueness of an equilibrium (Nash equilibrium) are well known, and alternative distributed algorithms, either synchronous or asynchronous, converging to the equilibrium (provided that it is unique) are available [1, 5].

The situation is more complicated for problems that are *not standard*, as the so called *vector* problems, where each user has a set of coefficients (degrees of freedom) to optimize. This happens for example, when each user is allowed to design its own linear precoder or to derive the optimal power allocation across a set of parallel subchannels. In this case, a unified framework is still missing.

Recently, Monderer and Shapley proposed in their seminal paper [6] a new class of games, called *potential games,* that was shown to be an interesting tool for studying non-standard problems. In [7] and [8], potential games were successfully applied to some *scalar* power control and interference avoidance problems in CDMA networks, respectively. In the game structure introduced in [6] and used in [7, 8], there is no coupling in the feasible strategies of the players, i.e., the set of the admissible players' strategies is given by the Cartesian product of the strategies' set of each player. Moreover, the results in [7] (and most of [6]) are valid only for scalar players' strategies.

However, in practice, many applications require a vector to optimize and impose a coupling among the feasible strategies of the players. For example, in the (scalar/vector) power control problems, the Quality of Service (QoS) requirements are, in general, formulated as constraints on the signal-to-interference plus noise ratio of each user [1, 3, 4]. Hence, the set of feasible strategies of each user depends on power allocations used by the others. In all these cases, the results of [7, 8, 6] cannot be used any more.

The scope of this paper is to overcome this issue, by generalizing the game structure of [6]. Our original contributions with respect to [7, 8, 6] are the following: 1) We introduce and fully characterize, in terms of existence and uniqueness of Nash Equilibrium (NE), different classes of strategic noncooperative potential games, where each player's strategy is a *vector* and the strategy set is *coupled*; 2) We provide alternative distributed algorithms based either on Gauss-Seidel or Jacobi scheme [5], along with their convergence to a NE of the game. After introducing this new framework, we show, by some examples, that a wide class of power control problems (standard and *non-standard*) with coupled constraints on the set of admissible users' powers can be efficiently solved using potential games. One of the proposed examples is also instrumental to point out an interesting and general relationship existing between the NEs of potential games and the equilibria of proper autonomous dynamic systems: A potential game can always be interpreted as an autonomous *gradient* dynamic system, whose Lyapunov function [9] is just the potential of the game. Such a dynamic system evolves toward an equilibrium point that represents one of the NEs of the potential game [10].

2. POTENTIAL GAMES

Consider a strategic non-cooperative game $\mathscr{G} = \{\Omega, \mathcal{X}, \{\Phi_q\}_{q\in\Omega}\}$, where Ω is the set of the Q players; $\mathcal{X} \subseteq \mathbb{R}^{mQ}$ is the set of pure strategies $\mathbf{x} \triangleq [\mathbf{x}_1^T, \ldots, \mathbf{x}_Q^T]^T \in \mathcal{X}$, with the m-length vector $\mathbf{x}_q$ representing the strategy of the q-th player; the function $\Phi_q : \mathcal{X} \mapsto \mathbb{R}$ is the payoff of the q-th player, which depends on the strategies $\mathbf{x}$ of *all* players. In the case of no coupling among players' strategies, the set $\mathcal{X}$ can be written as the Cartesian product of the strategy sets $\mathcal{X}_q$ of each player, i.e. $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_Q$. In the more general case of coupled constraints, each player q aims to restrict its strategy $\mathbf{x}_q$ to a subset of $\mathcal{X}_q$, denoted by $\mathcal{X}_q(\mathbf{x}_{-q}) \triangleq \{\mathbf{x}_q \in \mathcal{X}_q \mid (\mathbf{x}_q, \mathbf{x}_{-q}) \in \mathcal{X}\}$, that depends also on the strategies $\mathbf{x}_{-q}$ chosen by the other players.

We provide now the basic definitions and the main results of potential games, in the case of coupled strategy set. For the lack of space, in this paper we focus only on *exact*, and *ordinal* potential games. More general results are given in [10].

Definition 1 *A strategic game $\mathscr{G} = \{\Omega, \mathcal{X}, \{\Phi_q\}_{q\in\Omega}\}$ is called*

i) an exact *potential game if there exists a function $P : \mathcal{X} \mapsto \mathbb{R}$*

such that for all $q \in \Omega$ and $(\mathbf{x}_q, \mathbf{x}_{-q}), (\mathbf{y}_q, \mathbf{x}_{-q}) \in \mathcal{X}$:

$$\Phi_q(\mathbf{x}_q, \mathbf{x}_{-q}) - \Phi_q(\mathbf{y}_q, \mathbf{x}_{-q}) = P(\mathbf{x}_q, \mathbf{x}_{-q}) - P(\mathbf{y}_q, \mathbf{x}_{-q}); \quad (1)$$

ii) an ordinal *potential game if there exists a function $P : \mathcal{X} \mapsto \mathbb{R}$ such that for all $q \in \Omega$ and $(\mathbf{x}_q, \mathbf{x}_{-q}), (\mathbf{y}_q, \mathbf{x}_{-q}) \in \mathcal{X}$:*

$$\Phi_q(\mathbf{x}_q, \mathbf{x}_{-q}) - \Phi_q(\mathbf{y}_q, \mathbf{x}_{-q}) > 0 \Leftrightarrow P(\mathbf{x}_q, \mathbf{x}_{-q}) - P(\mathbf{y}_q, \mathbf{x}_{-q}) > 0. \quad (2)$$

Such a function P is called (exact, ordinal) potential *of the game*.

Through the whole paper, a game will be called potential if it has a potential function, according to either (1) or (2). In words, a strategic game is a potential game if there is a function that quantifies the difference in the payoff due to unilaterally deviating each player either exactly (exact potential game) or in sign (ordinal potential game). Note that the existence of a potential in a game does not directly guarantee the Pareto optimality of the NE. For example, some Cournot oligopolies are potential games with inefficient equilibria [6]. Rather than a warranty of Pareto efficiency, the potential function can be interpreted as a measure of the disagreement among the players, or, equivalently, of the drift toward the NE. In the jargon of dynamic system theory, the potential represents a Lyapunov function of the game, modelled as a dynamic system [10].

Potential functions are very useful tools for analyzing potential games, thanks to the following

Proposition 1 *Let $\mathscr{G} = \{\Omega, \mathcal{X}, \{\Phi_q\}_{q\in\Omega}\}$ be a potential game, with potential function P, and let $\tilde{\mathscr{G}} := \{\Omega, \mathcal{X}, \{P\}\}$ be the so-called coordination game, i.e. the game with the same structure of $\mathscr{G}$ but with all payoff functions replaced by P. Then, the set of NEs of $\mathscr{G}$ coincides with the set of NEs of $\tilde{\mathscr{G}}$.*

Proof. The proof follows directly from definitions (1) and (2). ■

Proposition 1 shows the importance for a game to have a potential function: we can study the properties of NEs and, more important, to find some equilibria, using a single function that does not depend on the particular player.

After recasting the study of properties of $\mathscr{G}$ into those of the corresponding coordination game $\tilde{\mathscr{G}}$, a natural question arises: *What relationship does exist among the NEs of $\tilde{\mathscr{G}}$ and the maxima of the potential function P on $\mathcal{X}$* ? The answer is given by the following

Theorem 2 ([10]) *Let $\mathscr{G} = \{\Omega, \mathcal{X}, \{\Phi_q\}_{q\in\Omega}\}$ be a potential game, with potential function P, and let $\mathcal{P}_{\max}$ denote the set of maxima of P on $\mathcal{X}$ (assumed non-empty). The following statements hold true: 1) If $\mathbf{x} \in \mathcal{P}_{\max}$, then $\mathbf{x}$ is a NE of $\mathscr{G}$. The converse, in general, is not true. But, if, in addition, $\mathcal{X}$ is a convex set with $\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_Q$ and P is a continuously differentiable function on the interior of $\mathcal{X}$, then: 2) If $\mathbf{x}$ is a NE of $\mathscr{G}$, then $\mathbf{x}$ is a stationary point of P; 3) Assume that P is concave on $\mathcal{X}$. If $\mathbf{x}$ is a NE of $\mathscr{G}$, then $\mathbf{x} \in \mathcal{P}_{\max}$. If P is strictly concave, such a NE must be unique.*

Thus, in the general case of a coupled strategy set, any maximum of P on $\mathcal{X}$ is a NE of the potential game (Theorem 2). Moreover, the set of NEs of the coordination game coincides with that of the potential game.[1] Therefore, the study of potential games can be carried out using two different approaches: 1) The classical framework of game theory (e.g., [11]) directly applied to the coordination game (Proposition 1); and 2) the framework of standard optimization theory applied to the potential function (Theorem 2). Both of these approaches make the analysis of potential games rather simple. For example, the existence of a NE for $\mathscr{G}$ comes directly from the existence of a maximum for the function P on the set $\mathcal{X}$. Moreover, most distributed algorithms coming from the optimization of scalar functions [5] can be applied, with some weak refinement, to the design of distributed algorithms converging to some NE of the potential game. Building on 1) and 2), we provide sufficient conditions for existence and uniqueness of the NE of a potential game and propose two different classes of iterative distributed algorithms that converge to a NE.

2.1. Existence and Uniqueness of Nash Equilibrium

A fundamental issue in strategic (infinite) noncooperative games is the study of the existence and uniqueness of pure strategy NE. In fact, not every strategic game admits a NE in pure strategy, and unfortunately only sufficient conditions for the existence are available. The study of the uniqueness is even worse, and, by now, only partial results are known [11]. For the special case of potential games, conditions for existence and uniqueness of NE can be obtained simply, as shown in the following.

Theorem 3 ([10]) *Let $\mathscr{G} = \{\Omega, \mathcal{X}, \{\Phi_q\}_{q\in\Omega}\}$ be a potential game, with a potential function P, and $\mathcal{P}_{\max}$ denoting the set of maxima of P (assumed non-empty). Then, $\mathscr{G}$ admits at least one NE.*

If, in addition, $\mathcal{X}$ is a compact, convex set, and P is a continuously differentiable function on the interior of $\mathcal{X}$ and strictly concave on $\mathcal{X}$, then the NE of $\mathscr{G}$ is unique.

Corollary 4 *Every finite potential game admits at least one NE [6].*

Corollary 5 *For infinite potential games, sufficient conditions for the existence of NE (non-emptyness of $\mathcal{P}_{\max}$) are that i) $\mathcal{X}$ is a compact set; and ii) P is upper semicontinuous on $\mathcal{X}$.*

It is interesting to observe that conditions for the uniqueness of NE obtained for uncoupled strategy set (Theorem 2) are valid also in the more general case of coupled strategies (Theorem 3).

2.2. Distributed Algorithms for Nash Equilibria

After deriving conditions for a potential game to have at least one NE, we need to design the rules that each player must follow to reach an equilibrium. To this end we assume that the same game could be myopically played repeatedly: in each round, every player has neither memory of past game-rounds nor speculation of future events, but it chooses its own strategy according to some decision rules that depend on the current state of the game. A new round of the same game is then played, until a NE of the game is reached. The main goal in the design of the algorithm is thus the choice of proper players' decision rules that guarantee the asymptotically stability of (at least some) NE of the game. We call these rules as stable decision rules and denote the set of stable rules for the q-th player by $\mathcal{D}_q(\mathbf{x}_q, \mathbf{x}_{-q})$, where we have explicitly shown the dependence of $\mathcal{D}_q$ on the players' strategies. We prove in [10] that, for example, the following decision rules are stable:

1) *Best response*:

$$D_q(\mathbf{x}_{-q}) = \left\{ \mathbf{x}_q^\star \in \mathcal{X}_q(\mathbf{x}_{-q}) : \mathbf{x}_q^\star = \arg\max_{\mathbf{x}_q} \Phi_q(\mathbf{x}_q, \mathbf{x}_{-q}) \right\}; \quad (3)$$

2) *Better response*:

$$D_q(\mathbf{x}) = \left\{ \mathbf{x}_q^\star \in \mathcal{X}_q(\mathbf{x}_{-q}) : \Phi_q(\mathbf{x}_q^\star, \mathbf{x}_{-q}) > \Phi_q(\mathbf{x}_q, \mathbf{x}_{-q}) \right\}; \quad (4)$$

3) *Gradient projection response*[2]*:*

$$D_q(\mathbf{x}) = \left[\mathbf{x}_q + \alpha_q \bigtriangledown_q \Phi_q(\mathbf{x}_q, \mathbf{x}_{-q})\right]_{\mathcal{X}_q(\mathbf{x}_{-q})}, \quad (5)$$

[1]Note that there exist other classes of potential games for which this correspondence does not holds true, as shown in [10].

[2]To use this rule the strategy set $\mathcal{X}$ must be compact and convex. Note that alternative gradient projections can be considered, as, e.g., that of [11].

where $[\mathbf{x}_q]_{\mathcal{X}_q(\mathbf{x}_{-q})}$ denotes the Euclidean projection of $\mathbf{x}_q$ on the set $\mathcal{X}_q(\mathbf{x}_{-q})$ and α_q is a sufficiently small positive number.

Given the stable decision rules, two different approaches can be followed in the choice of the iterative algorithms to be performed by the players, namely Gauss-Seidel and Jacobi based schemes.

In the Gauss-Seidel algorithm, the players update their strategy sequentially:

$$\mathbf{x}_q^{t+1} = \mathcal{D}_q\left(\mathbf{x}_1^{t+1}, \ldots, \mathbf{x}_{q-1}^{t+1}, \mathbf{x}_q^t, \mathbf{x}_{q+1}^t, \ldots, \mathbf{x}_Q^t\right), \qquad (6)$$

where $\mathcal{D}_q\left(\mathbf{x}_q, \mathbf{x}_{-q}\right)$ can be chosen among (3), (4), and (5).

In the Jacobi algorithm, all the players optimize their own strategies in a parallel fashion:

$$\mathbf{x}_q^{t+1} = \mathcal{D}_q\left(\mathbf{x}_1^{t}, \ldots, \mathbf{x}_{q-1}^{t}, \mathbf{x}_q^t, \mathbf{x}_{q+1}^t, \ldots, \mathbf{x}_Q^t\right), \qquad (7)$$

with $\mathcal{D}_q\left(\mathbf{x}_q, \mathbf{x}_{-q}\right)$ given by (5). In [10] we provide sufficient conditions for the convergence of the Gauss-Seidel update (6) with the decision rules (3)-(5) and of the Jacobi update (7) with the decision rule (5).

It is interesting to observe that, under proper assumptions on the potential function and the strategy set, the distributed optimization algorithms studied in [5] come out as a particular case of (6) and (7). For example, if the potential function P is continuous differentiable and strictly concave on $\mathcal{X}$, and $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_Q$, with each $\mathcal{X}_q$ closed and convex, algorithm (6) with $D_q(\mathbf{x})$ given by (3) coincides with the classical nonlinear Gauss-Seidel algorithm [5, Sec. 3.3.5], which is then guaranteed to converge to the maximum of P [5, Prop. 3.9], and thus to the unique NE of the game (Theorem 2).

3. TWO RELEVANT EXAMPLES

Many power control problems both in cellular and ad-hoc networks can be formulated as potential games and thus efficiently solved using the framework proposed in Section 2. Due to space limitation, in this paper we provide only two applications. A first example deals with standard power control problems in CDMA networks with coupled constraints. However, potential games also provide a useful tool to study *non-standard* problems, for which a unified framework is still missing, as shown in the second example. Such an example is also instrumental to understand the general relationship existing between the NEs of a potential game and the equilibrium points of a proper gradient dynamic system, having as a Lyapunov function the potential of the game [10].

3.1. Standard problems with coupled constraints

Consider a single-cell[3] CDMA system with total bandwidth W Hz and unspread bandwidth B Hz, supporting Q users. The signal-to-interference plus noise ratio (SINR) of the q-th user at the receiver is given by [1]

$$\mathrm{SIR}_q(\mathbf{p}) = \frac{W}{B}\frac{|h_q|^2 p_q}{1+\sum_{j\neq q}|h_j|^2 p_j}, \qquad (8)$$

where $\mathbf{p} = [p_1, \ldots, p_Q]^T$, p_q is the transmit power of the q-th user, $\{h_q\}_q$ is the set of channels from the users to to Base Station (BS), and the thermal noise power is normalized to one.

A general formulation for standard power control problems is

$$\begin{array}{ll} \text{minimize} & p_q \\ \text{subject to} & f_{iq}\left(\mathrm{SIR}_q(\mathbf{p})\right) \geq \gamma_{iq}, \quad i = 1, \ldots, n_q, \\ & p_q \in [0, P_q], q \in \Omega, \end{array} \qquad (9)$$

[3]The multi-cell case can be studied using the same approach [1].

where P_q is the total power budget of each transmitter, and $f_{iq}(\cdot)$ is a continuous function on $\mathbb{R}_+$. Observe that depending on the choice of $f_{iq}(\cdot)$'s, different types of QoS constraints from individual users, either in fixed/slow-fading or fast-fading Rayleigh channels, can be taken into account. For example, in the case of fixed or slow-fading Rayleigh channels, constraints on data rate, error probability, delay and outage probability of each user can be expressed as in (9), with a convex and compact feasible set [12]. In the case of fast-fading ergodic Rayleigh channels, constraints, e.g., on the maximum tolerable average BER, or minimum average throughput can be equivalently rewritten as a constraint on the lower bound of the average SINR of each user [13], with the same form as (9), still with a convex and compact feasible set.

However, QoS constraints give rise to a coupling among the individual powers, which makes problem (9) in general harder to study. Nevertheless, (9) can be equivalently formulated as the following strategic noncooperative game

$$\mathscr{G} = \left\{\Omega, \mathcal{X}, \{\log(p_q)\}_{q\in\Omega}\right\}, \qquad (10)$$

with

$$\mathcal{X} = \left\{\mathbf{p} \in \mathbb{R}_+^Q : f_{iq}\left(\mathrm{SIR}_q(\mathbf{p})\right) \geq \gamma_{iq},\ p_q \leq P_q,\ \forall i, \forall q \in \Omega\right\}.$$

It is not difficult to verify that the game $\mathscr{G}$ as defined in (10) is an *exact* potential game with potential function[4]

$$P(\mathbf{p}) = \sum_{q\in\Omega}\log(p_q). \qquad (11)$$

Observe that, if the original problem (9) is feasible, then the set $\mathcal{X}$ defined in (10) is compact. Moreover, multiple equilibria may exist. If, in addition, the set is also convex, then the NE of the game (and thus the solution of (9)) is unique (Theorem 3) and it can be reached using the algorithms given in the Section 2.2.

3.2. Non-standard problem

Consider again system model (8), and now the strategic noncooperative game

$$\mathscr{G} = \left\{\Omega, \{\mathcal{X}_q\}_{q\in\Omega}, \{\Phi_q\}_{q\in\Omega}\right\}, \qquad (12)$$

with $\mathcal{X}_q = \mathbb{R}_+$, and $\Phi_q(\mathbf{p}) = \log(1 + \mathrm{SIR}_q(\mathbf{p})) - c_q(p_q)$, where $\mathrm{SIR}_q(\mathbf{p})$ is defined in (8), and $c_q(p_q)$ denotes a pricing function (assumed to be twice continuously differentiable, nondecreasing, and strictly convex in p_q). The pricing function may serve different purposes such as a control employed by the BS to limit the transmit power of each user and thus the interference generated, or as a penalty in terms of battery usage. Note that $c_q(p_q)$ contains, as a particular case, the linear function largely used as pricing in the literature (see, e.g. [14]). Observe that, because of the pricing factor, at NE some user may not transmit at all [14]. This solution occurs if the price factor is set too high, so that some user prefers to not transmit, depending on its channel gain and interference. In order to avoid this possibility, a constraint on the lower bound of each $\mathrm{SIR}_q(\mathbf{p})$ in (8) can be introduced. In this case, the admissible set of the game becomes coupled. As a similar problem has already considered in (9), we assume in the following a feasible set $\mathcal{X}_q$ as defined in (12).

The game $\mathscr{G}$ in (12) does not fall within the class of standard problems of [1], because the best response strategy of each player does not verify the monotonicity property, required for a function to

[4]The choice of the log function in (11) is only instrumental to have a strictly concave potential, so that Theorem 3 can be applied.

be standard. Nevertheless, the full characterization of the game can still be obtained, using the framework of potential games. In fact, it is straightforward to see that $\mathscr{G}$ is an *exact* potential game, with potential function

$$P(\mathbf{p}) = \log\left(1 + \sum_{q\in\Omega} |h_q|^2 p_q\right) - \sum_{q\in\Omega} c_q(p_q). \tag{13}$$

Moreover, since the function $P(\mathbf{p})$ in (13) is strictly concave in $\mathbf{p}$ and the strategy set $\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_Q$ is compact[5] and convex, the game admits a unique NE (Theorem 3). Note that this result generalizes [14], where the uniqueness of NE is proved only for linear pricing and under a constraint on the maximum number of users.

Alternative distributed algorithms can be used to reach the equilibrium. The Gauss-Seidel algorithm, with, e.g., best response as in (3) is

$$p_q^{i+1} = \arg\max_{p_q \geq 0} P\left(p_1^{i+1}, \ldots, p_{q-1}^{i+1}, p_q, p_{q+1}^{i}, \ldots, p_Q^{i}\right). \tag{14}$$

The Jacobi iterative algorithm is instead given by

$$p_q^{i+1} = p_q^i + \alpha\left[\frac{|h_q|^2}{1 + \sum_{j\in\Omega} |h_j|^2 p_j^i} - \frac{d}{dp_q}\, c_q(p_q)|_{p_q = p_q^i}\right]^+, \tag{15}$$

where $[x]^+ = \max(0, x)$. The convergence of Gauss-Seidel algorithm and Jacobi algorithm (with a proper choice of α) can be proved using the framework of potential games [10], or, for this particular case using directly [5, Prop. 3.9] and [11, The. 9], respectively.

To compare the performance of the two algorithms, we plot in Figure 3.2, as an example, the evolution of the powers with the iterations using (14) and (15), with pricing function $c_q(p_q) = \gamma |h_q|^2 (p_q)^2$. For the sake of comparison, we also report the globally optimal power allocation, obtained using a centralized optimization. Observe that a Guss-Seidel algorithm is faster that Jacobi algorithm. However, thanks to the strict concavity of (13), both algorithms convergence to the globally optimal solution (Theorem 2).

The game $\mathscr{G}$ introduced in (12) is also instrumental to show the interplay existing between potential games and dynamic systems, as we argue next. Consider the following autonomous gradient dynamic system

$$\frac{d}{dt} p_q(t) = \alpha\left[\frac{\partial}{\partial p_q} P(\mathbf{p}(t))\right]^+, \quad q \in \Omega, \quad t \geq 0, \tag{16}$$

with $p_q(0) \geq 0$ and $P(\mathbf{p})$ given by (13). Observe that the Jacobi algorithm in (15) is just the discrete-time approximation of the differential equation (16), for a sufficiently small α. Furthermore, the equilibrium of the dynamic system (16), (i.e. the point where $dp_q(t)/dt = 0$ for all q), corresponds to the (unique) NE of the game defined in (12) (Theorem 2). More interesting, the dynamic system converge to this equilibrium, from any initial condition, and a Lyapunov function for this system [9] is just the (scaled version of) potential function P given in (13). In fact, consider the following candidate Lyapunov function

$$V(\mathbf{p}) = P_{\max} - P(\mathbf{p}), \tag{17}$$

where $P_{\max} \triangleq P(\mathbf{p}^\star)$ denotes the (unique) maximum of P over $\mathcal{X}$ defined in (12). It is straightforward to see that $V(\mathbf{p}) \geq 0\ \forall \mathbf{p} \in \mathcal{X}$ and $V(\mathbf{p}) = 0$ if and only if (iff) $\mathbf{p} = \mathbf{p}^\star$. Furthermore, $V(\mathbf{p})$ is nondecreasing along the trajectories of the system (16), since

$$\begin{aligned}\frac{d}{dt} V(\mathbf{p}(t)) &= -\nabla_{\mathbf{p}}^T P(\mathbf{p})\left(\frac{d}{dt}\mathbf{p}(t)\right) = \\ &= -\alpha \left\|[\nabla_{\mathbf{p}} P(\mathbf{p})]^+\right\|^2 \leq 0,\end{aligned}$$

with $\frac{d}{dt} V(\mathbf{p}(t)) = 0$ iff $\left[\nabla_{\mathbf{p}}^T P(\mathbf{p})\right]^+ = 0$, i.e. iff $\mathbf{p} = \mathbf{p}^\star$. It follows that (17) is a valid Lyapunov function for the dynamic system (16). Lasalle's invariance principle [9] asserts that if there exists a Lyapunov function that is negative semidefinite[9] along the trajectories of the dynamic system, then solutions, originating in the compact set $\mathcal{X} = [0, P]^Q$ converge to the largest invariant set[9] in

$$\mathcal{D} = \left\{\mathbf{p} \in \mathcal{X} : \frac{d}{dt} V(\mathbf{p}(t)) = 0\right\} \tag{18}$$

As $\mathcal{D}$ in (18) contains only the unique NE of the game $\mathscr{G}$ in (12), dynamic system (16) converges to such an equilibrium. This proves also the convergence of the Jacobi algorithm (15) to the equilibrium (for a sufficiently small α).

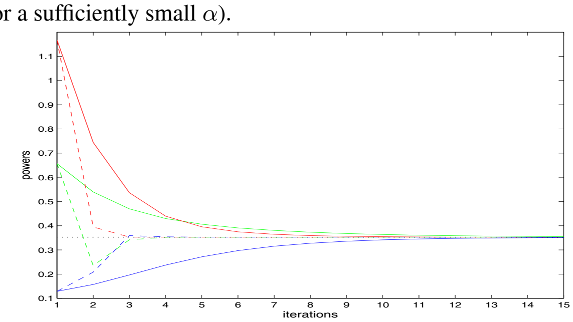

Fig. 1. Powers of three users vs. iterations. Dashed, solid and dotted line curves refer on Gauss-Seidel algorithm in (14), Jacobi algorithm in (15) and the globally optimal centralized solution, respectively; $\alpha = 0.1$, $\gamma = 0.5$.

[5] Note that, since at the NE the set of optimal powers $\mathbf{p}_1^\star, \ldots, \mathbf{p}_Q^\star$ is finite, we can always replace in the game $\mathscr{G}$ defined in (12) the original set $\mathcal{X}_q = \mathbb{R}_+$ with the closed and bounded set $\mathcal{X}_q = [0, P]$, where P is chosen so that $p_q^\star < P, \forall q \in \Omega$.

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