A Scalable Collusion-Resistant Multi-Winner Cognitive Spectrum Auction Game

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***Abstract*—Dynamic spectrum access (DSA), enabled by cognitive radio technologies, has become a promising approach to improve efficiency in spectrum utilization, and the spectrum auction is one important DSA approach, in which secondary users lease some unused bands from primary users. However, spectrum auctions are different from existing auctions studied by economists, because spectrum resources are interference-limited rather than quantity-limited, and it is possible to award one band to multiple secondary users with negligible mutual interference. To accommodate this special feature in wireless communications, in this paper, we present a novel multi-winner spectrum auction game not existing in auction literature. As secondary users may be selfish in nature and tend to be dishonest in pursuit of higher profits, we develop effective mechanisms to suppress their dishonest/collusive behaviors when secondary users distort their valuations about spectrum resources and interference relationships. Moreover, in order to make the proposed game scalable when the size of problem grows, the semi-definite programming (SDP) relaxation is applied to reduce the complexity significantly. Finally, simulation results are presented to evaluate the proposed auction mechanisms, and demonstrate the complexity reduction as well.**

***Index Terms*—Cognitive radio, spectrum auction, collusion-resistant mechanism, scalable algorithm.**

I. INTRODUCTION

As the demand for wireless spectrum has been growing rapidly with the deployment of new wireless applications and devices in the last decade, the regulatory bodies such as the Federal Communications Commission (FCC) have begun to consider more flexible and comprehensive usage of available spectrum [1][2]. With the development of cognitive radio technologies [3], dynamic spectrum access becomes a promising approach, which allows unlicensed users (secondary users) dynamic, opportunistic access to the licensed bands owned by legacy spectrum holders (primary users) in either a non-cooperative fashion [4]–[6] or a cooperative fashion [7]–[13].

In the non-cooperative approach, secondary users' existence is transparent to primary users, and secondary users have to frequently sense the radio environment to detect the presence of primary users. Whenever finding a spectrum opportunity when the primary user is absent, secondary users are allowed to occupy the spectrum; but they must immediately vacate the band when the primary user appears. Several schemes have been previously proposed in the literature. For instance, in [4], the authors devised rules for secondary users to utilize available spectrum while avoiding interference with their neighbors based on a graph-theoretic model. The work in [5] examined the secondary users' access patterns to propose a feasible spectrum sharing scheme. In [6], the authors proposed a primary prioritized Markovian dynamic spectrum access scheme to optimally coordinate secondary users' spectrum access and achieve a good statistical tradeoff between efficiency and fairness.

However, imperfect spectrum sensing may lead to missed spectrum opportunities as well as collision with primary users. To circumvent the difficulties, an alternative is the cooperative approach where spectrum opportunities are *announced* by primary users rather than *discovered* by secondary users. Since primary users have the incentive to trade their temporarily unused bands for monetary gains and secondary users want to lease some bands for data transmission, they may negotiate the price for a short-term lease. With the emerging applications of mobile ad hoc networks envisioned in civilian usage, it is reasonable to assume secondary users are selfish and aim at maximizing their own interests because they do not serve a common goal or belong to a single authority. Operated by human or service providers, they are also capable of acting intelligently. Based on these assumptions, there are several previous efforts studying dynamic spectrum access via pricing and auction mechanisms. In [7], the price of anarchy, i.e., the loss due to the lack of a central authority, was analyzed for spectrum sharing in WiFi networks. In [8], a demand responsive pricing framework was proposed to maximize the profits of legacy spectrum operators while considering the buyers' response model. An auction-based mechanism was proposed in [9] to efficiently share spectrum among secondary users in interference-limited systems. In [10], the authors considered a multi-unit sealed-bid auction for efficient spectrum allocation. In [11], a real-time spectrum auction framework with interference constraints was proposed to get a conflict-free allocation. In [12] [13], a belief-assisted distributive pricing algorithm was proposed to achieve efficient dynamic spectrum allocation based on double auction mechanisms. Although existing schemes have enhanced spectrum allocation efficiency

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through market mechanisms, some critical challenges still remain unanswered.

First, in most of the current auctions, one licensed band (or a collection of multiple bands) is awarded to a unique winner just like traditional auctions studied by economists [14]. However, the spectrum resource is quite different from other commodities in that it is *interference-limited* rather than *quantity-limited*, because it is reusable by wireless users geographically far apart. In some application scenarios where secondary users only need to communicate within a short range, such as a wireless personal area network (WPAN) centered around a person's workspace, the transmission power is quite low, and hence even users with moderate distances can simultaneously access the same band. In this case, allowing multiple winners to lease the band is an option consented by everyone: primary users get higher revenue, secondary users get more chances to access the spectrum, and spectrum usage efficiency gets boosted as well from the system designer's perspective. To the best of our knowledge, such an auction does not exist in the literature, and we coin the name *multi-winner auction* to highlight the special features of the new auction game, in which auction outcomes (e.g., the number of winners) highly depend on the geographical locations of the wireless users.

Second, although a few papers (e.g., [9][11]) have discussed spectrum auctions under interference constraints, most of them are based on the assumption that secondary users are truth-tellers, that is, they will honestly reveal their private information such as the valuations and interference relationships. However, since secondary users are selfish by nature, they may misrepresent their private information in order for a higher payoff. Therefore, proper mechanisms have to be developed to provide incentives to reveal true private information. Although the Vickery-Clarke-Groves (VCG) mechanism is a possible choice enforcing that users bid their true valuations [15], it is also well-known to suffer from several drawbacks such as low revenue [16][17]. As auction rules significantly impact bidding strategies, it is of essential importance to develop new auction mechanisms to overcome the disadvantages.

Third, mechanisms to be developed should take into consideration the collusive behavior of selfish users, which is a prevalent threat to efficient spectrum utilization but has been generally overlooked [13]. Driven by their pursuit of higher payoffs, a clique of secondary users may cheat together, and sometimes they may even have a more facilitated way to exchange information for collusion if they belong to the same service provider. Furthermore, awarding the same band to multiple buyers simultaneously under interference constraints, the multi-winner auction makes possible new kinds of collusion [18], besides the bidding ring collusion[1] from traditional auctions. Emerging kinds of collusion will be discussed in detail later in this paper, and effective countermeasures have to be developed against them.

Last but not least, it is much more meaningful to show the proposed scheme can be applied in practice, where complexity issues come into the spotlight: the mechanism should be easy to implement, and it should be scalable when more and more users are incorporated into the auction game. However, as we analyze later in this paper, the optimal resource allocation that maximizes the system utility in the auction is an NP-complete problem [19] whose exact solution needs a processing time increasing exponentially with the size of the problem, and hence the computational complexity becomes too formidable to be practical when the number of users is large. By applying the semi-definite programming (SDP) relaxation [20] to the original problem, a tight upper bound can be obtained in polynomial time.

The rest of this paper is organized as follows. In Section II, the model for a multi-winner cognitive spectrum auction is described, and several kinds of collusion are illustrated. In Section III, we develop auction mechanisms that not only yield high revenue but also prevent user collusion, and employ the SDP relaxation to make the scheme implementable and scalable. The one-band auction game is generalized to a multi-band auction in Section IV. Simulation results are presented in Section V, and Section VI concludes the paper.

Notations: $\mathbf{A} \in \mathscr{M}^{m\times n}$ means $\mathbf{A}$ is a matrix with dimension $m \times n$, and $\mathbf{b} \in \mathscr{M}^{m\times 1}$ indicates $\mathbf{b}$ is a column vector with length m. Denote their entries as A_{ij} and b_i, respectively. The trace of a matrix $\mathbf{A}$ is denoted by $\text{tr}(\mathbf{A})$, and its rank is denoted by $\text{rank}(\mathbf{A})$. The 2-norm of a vector $\mathbf{b}$ is denoted by $|\mathbf{b}|_2$. The all-zero, all-one, and identity matrices are denoted by $\mathbf{O}$, $\mathbf{1}$, and $\mathbf{I}$, respectively, and their dimensions are given in the subscript when there is room for confusion. $\mathbf{S} \in \mathscr{S}^n$ means $\mathbf{S}$ is an $n \times n$ real symmetric matrix, and $\mathbf{S} \succeq \mathbf{O}$ implies $\mathbf{S}$ is positive semi-definite. The Kronecker product of two matrices $\mathbf{A}$ and $\mathbf{B}$ is denoted by

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} A_{11}\mathbf{B} & A_{12}\mathbf{B} & \cdots & A_{1n}\mathbf{B} \\ A_{21}\mathbf{B} & A_{22}\mathbf{B} & \cdots & A_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}\mathbf{B} & A_{m2}\mathbf{B} & \cdots & A_{mn}\mathbf{B} \end{bmatrix}. \quad (1)$$

Denote $\mathbf{b}_{-i} = [b_1, b_2, \ldots, b_{i-1}, b_{i+1}, \ldots, b_m]^T$ as a new vector with the ith entry of $\mathbf{b}$ excluded. Similarly, if W is a set of indices, $\mathbf{b}_{-W}$ implies all the entries whose indices fall in W are removed. $|W|$ denotes the cardinality of a set W. For two sets W_1 and W_2, the set difference is defined as $W_1 \setminus W_2 = \{x | x \in W_1 \text{ and } x \notin W_2\}$.

II. SYSTEM MODEL

We consider a cognitive radio network where N secondary users coexist with M primary users, and primary users seek to lease their unused bands to secondary users for monetary gains. We model it as an auction where the sellers are the primary users, the buyers are the secondary users, and the auctioneer is a spectrum broker who helps coordinate the auction. Assume there is a common channel to exchange necessary information and a central bank to circulate money in the community. For simplicity, we assume each primary user owns one band exclusively, and each secondary user needs only one band. In this paper, we first consider the auction with a single band ($M = 1$), and later extend it to the multi-band auction.

[1] In the bidding ring collusion, several potential buyers form a bidding ring by making an agreement not to outbid one another, which may keep prices low and decrease the seller's revenue. It can be eliminated by setting up an optimal reserve price [13].

The system designer determines a fixed leasing period T according to channel dynamics and overhead considerations, that is, the duration should be short enough to make spectrum access flexible but not too short since the overhead of the auction would become problematic. At the beginning of each leasing period, if a primary user decides not to use his/her own licensed band for the next duration of T, he/she will notify the spectrum broker of the intention to sell the spectrum rights. Meanwhile, the potential buyers submit their sealed bids $\mathbf{b} = [b_1, b_2, \ldots, b_N]^T$ to the spectrum broker simultaneously, where b_i is the bid made by user i. According to the bids and channel availability, the broker decides both the allocation $\mathbf{x} = [x_1, x_2, \ldots, x_N]^T$ and the prices $\mathbf{p} = [p_1, p_2, \ldots, p_N]^T$, where $x_i = 1$ means secondary user i wins some band, $x_i = 0$ otherwise, and p_i is the price of the band for the ith secondary user[2]. Alternatively, we can define the set of winners as $W \subseteq \{1, 2, \ldots, N\}$, where $i \in W$ if and only if $x_i = 1$. Assume user i gains value v_i from transmitting information in the leased band, his/her reward is

$$r_i = v_i x_i - p_i, \; i = 1, 2, \ldots, N. \tag{2}$$

Given all users' valuations $\mathbf{v} = [v_1, v_2, \ldots, v_N]^T$, the system utility, or the *social welfare*[3], can be represented by

$$U_{\mathbf{v}}(\mathbf{x}) = \sum_{i=1}^{N} v_i x_i = \sum_{i \in W} v_i. \tag{3}$$

Since the proposed multi-winner auction awards the band simultaneously to several secondary users according to their mutual interference, interference plays an important role in the auction. There are several models for wireless interference [21]–[24], such as the protocol model and the physical model. In this paper, we will mainly focus on the well-known protocol model [21] which is simpler to understand, in order to highlight our contributions in auction mechanisms. With the protocol model employed, mutual interference in Fig. 1 (a) where $N = 6$ secondary cognitive base stations compete for the spectrum lease can be well captured by a conflict graph (Fig. 1 (b)), or equivalently, by an $N \times N$ adjacency matrix $\mathbf{C}$ (Fig. 1 (c)). By collecting reports from secondary users about their locations or their neighbors, the spectrum broker keeps the matrix $\mathbf{C}$ updated, even if the interference constraints change from time to time because of the slow movement of secondary users. When $C_{ij} = 1$, user i and user j cannot access the same band simultaneously, and if they do, neither of them gains due to collision. Therefore, the interference constraint is $x_i + x_j \leq 1$ if $C_{ij} = 1$.

However, our method can also be extended to the physical model in [23] which describes interference in a more accurate way but is more complicated. Under the physical model, only transmissions with the received signal-to-interference-and-noise ratio (SINR) exceeding some threshold β are considered

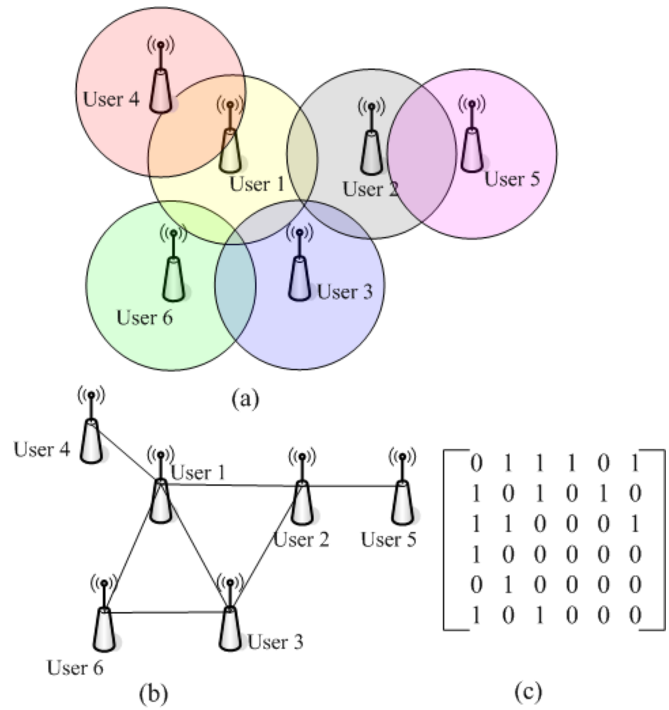

Fig. 1. Illustration of the interference structure in a cognitive spectrum auction. (a) physical model; (b) graph representation; (c) matrix representation.

successful, i.e., $g_{ii}P / \left(\sum_{j \neq i} g_{ji} P x_j + Z_i\right) \geq \beta$, where g_{ji} represents the channel gain from jth user's transmitter to ith user's receiver, Z_i is the noise at receiver i, and we assume all users use the same power P. By neglecting the noise term when interference is the dominant factor in the system, the condition for simultaneous transmissions when no individual is impaired by mutual interference can be further reduced to $\sum_{j=1}^{N} \alpha_{ji} x_j \leq 0$ if $x_i = 1$, where we define $\alpha_{ii} = -1$ and $\alpha_{ji} = \beta g_{ji} / g_{ii}, i \neq j$. We will briefly discuss auction mechanisms under the physical model at the end of Section III.

The special feature that multiple winners can share one band results in new kinds of potential collusion. In the following, we illustrate them by simple examples in the situation of Fig. 1, assuming users $\{2, 4, 6\}$ are supposed to be winners in absence of collusion.

- *Loser collusion*. A group of losers without mutual interference, e.g., users $\{1, 5\}$, cannot win the band because the band is less worth to them than to the winners. However, by collusively raising their bids beyond their valuations, the group may beat the winners and win the band instead. If the prices charged to them are still lower than their true valuations even if they overbid, they will receive positive payoffs from this kind of collusion. Hence, auction mechanisms should prevent overbidding colluders from gaining profits.
- *Sublease collusion*. After the auction, some of the winners may decide not to access the spectrum but sublease to other secondary users who failed to win the band in the auction. For instance, users $\{2, 6\}$ may negotiate a price with potential buyers $\{3, 5\}$ and sublease the band as long as all of them agree on the sublease price. Earning

[2]Different from the sequential auction which lasts for multiple rounds, in a sealed bid auction where buyers submit their bids simultaneously only once, the "pay-as-bid" strategy cannot enforce truth-telling. Hence, the price p_i is upbounded by his/her bid b_i, and determined by the auction mechanism.

[3]The social welfare measures the system-wide utility created by the transaction of commodities in the auction. Since prices paid by buyers and revenue gained by sellers cancel each other out, the social welfare does not depend on prices $\{p_i\}$.

extra profits effortlessly, colludes take away some benefits which should be credited to the primary user.

- *Kick-out collusion*. Several users belonging to the same group of interests attempt to manipulate the auction outcome by misrepresenting mutual interference. Assume users $\{4,5,6\}$ form such a group. Now, user 4 and user 6 will claim they have mutual interference with users 2, i.e., $\hat{C}_{42}=1$ and $\hat{C}_{62}=1$, to kick user 2 out, and welcome their ally, user 5, to join in the winner set.

III. One-Band Multi-Winner Auction

Defining rules for winner determination and price determination, mechanism design plays an important role in an auction, since it greatly affects the auction outcome as well as user behavior. For example, the widely employed VCG mechanism [15] ensures the maximum system utility and enforces that all buyers bid their true valuations in the absence of collusion, i.e., $b_i = v_i \ (i = 1, 2, \ldots, N)$. Although the VCG mechanism could be applied to the multi-winner auction, serious drawbacks such as low revenue and vulnerability to user collusion make it less attractive, as shown in [18] through specific examples in cognitive spectrum auctions. Therefore, we need to develop suitable mechanisms for the multi-winner auction which guarantee system efficiency, yield high revenue, prevent potential collusion, and are easy to implement.

A. The Optimal Allocation

Because the goal of dynamic spectrum access is to improve the efficiency of spectrum utilization, the auction mechanisms should be designed such that the social welfare is maximized, that is, the band is awarded to the secondary users who value them most.

In a cognitive spectrum auction, only those without mutual interference can be awarded the band simultaneously, and we group them together as *virtual bidders*, whose valuations equal the sum of the individual valuations. Take Fig. 1 for example, there are seventeen virtual bidders, such as $\{1\}$, $\{1,5\}$, $\{4,5,6\}$ and so on; on the other hand, combinations like $\{1,3\}$ and $\{2,5,6\}$ are not virtual bidders due to interference. In order to achieve full efficiency, the virtual bidder with the highest bid will win the band. It is unnecessary to list all virtual bidders explicitly; instead, the optimal allocation $\mathbf{x}$ can be determined by the following N-variable binary integer programming (BIP) problem,

$$\begin{aligned} U^*_{\mathbf{v}} = \max_{\mathbf{x}} \ & \sum_{i=1}^{N} v_i x_i, \\ \text{s.t.} \ & x_i + x_j \le 1, \ \forall i,j \text{ if } C_{ij}=1, \ \text{(interference constraints)} \\ & x_i = 0 \text{ or } 1, \ i = 1,2,\ldots,N, \end{aligned} \tag{4}$$

where interference constraints require that secondary users with mutual interference should not be assigned the band simultaneously.

B. Collusion-Resistant Pricing Strategies

After introducing the concept of virtual bidders, the multi-winner spectrum auction becomes similar to the single-winner auction, and hence it is possible to employ the second-price strategy[4]. By applying the second-price mechanism to the auction consisting of virtual bidders, the virtual bidder with the highest bid wins the band (ties are broken randomly if two virtual bidders have the same valuation), and pays the highest bid made by the virtual bidder only consisting of losers. This can be done by solving two optimal allocation problems in succession. First, we solve (4) to determine the set of winners W, or the virtual winner. Then, we remove all the winners W from the system, and solve the optimization problem again to calculate the maximum utility, denoted by $U^*_{\mathbf{v}_{-W}}$, which is the amount of money that the virtual winner has to pay.

We have to point out that the new pricing strategy sacrifices the enforcement of truth-telling a little bit for higher revenue and more robustness against collusion; however, since the proposed pricing strategy is quite similar to the second-price mechanism where users bid their true valuations, we expect users will not shade their bids too much from their true valuations. Thus, we neglect the difference between b_i and v_i in the following analysis to focus on revenue and robustness aspects of the new mechanisms.

The remaining problem is splitting the payment $U^*_{\mathbf{v}_{-W}}$ among the secondary users within the virtual winner. This is quite similar to a Nash bargaining game [25] where each selfish player proposes his/her own payment during a bargaining process such that the total payment equals $U^*_{\mathbf{v}_{-W}}$, and it is well-known that the Nash bargaining solution (NBS), which maximizes the product of individual payoffs, is an equilibrium [25]. In the proposed auction, no individual bargaining is necessary; instead, the spectrum broker directly sets the NBS prices for each winner, and everyone is ready to accept them since they are equilibrium prices. The pricing strategy is the solution to the following optimization problem,

$$\begin{aligned} \max_{\{p_i \in [0, v_i],\, i \in W\}} & \prod_{i \in W} (v_i - p_i), \\ \text{s.t.} & \sum_{i \in W} p_i = U^*_{\mathbf{v}_{-W}}. \end{aligned} \tag{5}$$

Proposition 1: User i has to pay the price

$$p_i = \max\{v_i - \rho, 0\}, \ \text{for } i \in W, \tag{6}$$

where ρ is chosen such that $\sum_{i \in W} p_i = U^*_{\mathbf{v}_{-W}}$. In particular, if $\hat{p}_i \triangleq v_i - \frac{U^*_{\mathbf{v}} - U^*_{\mathbf{v}_{-W}}}{|W|} \ge 0$ for any i, $p_i = \hat{p}_i$ will be the solution.

Proof: See Appendix A. ■

It can be seen that the payment is split in such a way that the profits are shared among the winners as equally as possible. Different from the VCG pricing strategy which sometimes may yield low revenue or even zero revenue, such a pricing strategy always guarantees that the seller receives revenue as much as $U^*_{\mathbf{v}_{-W}}$. Moreover, if some losers collude to beat the winners by raising their bids, they will have to pay more than $U^*_{\mathbf{v}_{-W}}$; however, the payment is already beyond what the

[4]In a second-price auction, the bidder with the highest bid wins the commodity, and pays the amount of money equal to the second highest bid. It is well-known that submitting bids equal to their true valuations is the dominant strategy [14], since the buyer may end up with paying more money than what it is actually worth if he/she submits a bid higher than the true valuation, and may lose the opportunity to win if he/she submits a lower one.

band is actually worth to them, and as a result, loser collusion is completely eliminated. Nevertheless, users can still benefit from the sublease collusion, and hence we call the pricing strategy in (5) the *partially collusion-resistant pricing strategy*.

In order to find a *fully collusion-resistant pricing strategy*, we have to analyze how sublease collusion takes place, and add more constraints accordingly. It happens when a subset of the winners $W_C \subseteq W$ subleases the band to a subset of the losers $L_C \subseteq L$, where $L = \{1, 2, \ldots, N\} \setminus W$ denotes the set of all losers. The necessary condition for the sublease collusion is $\sum_{i \in W_C} p_i < \sum_{i \in L_C} v_i$, so that they can find a sublease price in between acceptable to both parties. Given any colluding-winner subset $W_C \subseteq W$, the potential users who may be interested in subleasing the band should have no mutual interference with the remaining winners $W \setminus W_C$; otherwise, the band turns out to be unusable. Denote the set of all such potential users by $L(W \setminus W_C)$, i.e., $L(W \setminus W_C) \triangleq \{i \in L | C_{ij} = 0, \forall j \in W \setminus W_C\}$. Therefore, as long as prices are set such that $\sum_{i \in W_C} p_i \geq \max_{L_C \in L(W \setminus W_C)} \sum_{i \in L_C} v_i$, there will be no sublease collusion. Note that $\max_{L_C \in L(W \setminus W_C)} \sum_{i \in L_C} v_i$ is the maximum system utility $U^*_{\mathbf{v}_{L(W \setminus W_C)}}$ which can be obtained by solving the optimal allocation problem within the user set $L(W \setminus W_C)$, thus the optimum collusion-resistant pricing strategy is the solution to the following problem,

$$\max_{\{p_i \in [0, v_i], i \in W\}} \prod_{i \in W} (v_i - p_i), \quad \text{s.t.} \sum_{i \in W_C} p_i \geq U^*_{\mathbf{v}_{L(W \setminus W_C)}}, \forall W_C \subseteq W. \tag{7}$$

When $W_C = W$, the constraint reduces to $\sum_{i \in W} p_i \geq U^*_{\mathbf{v}_{-W}}$, which incorporates the constraint in (5) as a special case. There are $2^{|W|} - 1$ constraints in total because each of them corresponds to a subset $W_C \subseteq W$ except $W_C = \emptyset$. From another perspective, this actually takes into consideration of virtual bidders consisting of both winners and losers, in contrast to the previous pricing strategy where only those consisting of losers are considered.

C. Interference Matrix Disclosure

So far, our auction mechanism is based on the assumption that the underlying interference matrix $\mathbf{C}$ reflects the true mutual interference relationships between secondary users. However, since $\mathbf{C}$ comes from secondary users' own reports, it is quite possible that the selfish users manipulate this information just as what they may do with their bids. If cheating could help a loser become a winner, or help a winner pay less, the selfish users would have incentives to do so, which would compromise the efficiency of the spectrum auction. Also, the cheating behavior may happen individually or in a collusive way. Therefore, we have to carefully consider whether they have such an incentive to deviate, and if so, how to fix the potential problem. Here, we assume there are no malicious users who determine to do harm to others or even the whole system.

In order to obtain the matrix $\mathbf{C}$, the spectrum broker has to collect information from secondary users. Secondary users may report their locations in terms of coordinates, and the spectrum broker calculates the matrix according to their distances. In this way, secondary users do not have much freedom to fake an interference relationship in favor of themselves. Alternatively, secondary users may directly inform the spectrum broker about who are their neighbors, and hence they are able to manipulate the matrix, either by concealing an existing interference relationship or by fabricating an interference relationship that actually does not exist.

When secondary users have little information about others, they will misrepresent the interference relationships only if they do not get punished, even in the worst case. Assume user j lies about C_{jk}. When users j and k do not mutually interfere, i.e., $C_{jk} = 0$, but user j claims $\hat{C}_{jk} = 1$, he/she may lose an opportunity of being a winner since an extra interference constraint is added; on the other hand, if $C_{jk} = 1$ but user j claims $\hat{C}_{jk} = 0$, user j may end up winning the band together with user k, but the band cannot be used at all due to strong interference. In short, the worst-case analysis suggests secondary users have no incentive to cheat whenever information is limited.

When secondary users somehow have more information about others, they may distort the information in a more intelligent way, that is, they can choose when to cheat and how to cheat. Nevertheless, by investigating whether user j is better off by misrepresenting C_{jk}, we show that truth-telling is an equilibrium from which no individual would have the incentive to deviate unilaterally. We discuss all possible situations in what follows.

1) Under the condition that user j is supposed to be a loser.
   1a. Claim $\hat{C}_{jk} = 1$ against the truth $C_{jk} = 0$. By doing this, user j actually introduces an additional interference constraint to himself/herself, but since user j is already a loser, nothing would change.
   1b. Claim $\hat{C}_{jk} = 0$ against the truth $C_{jk} = 1$. Removing a constraint possibly helps user j to become a winner, but in the case, user k is also one of the winners. Then, user j has to pay a band that turns out to be unusable due to strong mutual interference with user k. This is unacceptable to user j.
2) Under the condition that user j is supposed to be a winner.
   2a. Claim $\hat{C}_{jk} = 0$ against the truth $C_{jk} = 1$. If user j is the only one among the winners that has interference with user k, it would take user k into the winner set, which would in turn make user j suffer from mutual interference.
   2b. Claim $\hat{C}_{jk} = 1$ against the truth $C_{jk} = 0$. If user k is not a winner, doing this would change nothing. If user k is indeed a winner, user j takes the risk of throwing himself/herself out of the winner set. Even if user j has enough information to secure he/she can still be a winner, kicking out user k does not necessarily make user j pay less.

In sum, no individual has the incentive to cheat even if there is enough information to make the intelligent cheating possible.

Similar analysis can be applied to the situation where a group of secondary users are able to distort the information collusively, and we find that kick-out collusion defined in

Section II is the only way that colluders gain an advantage. If channels are symmetric, i.e., $C_{jk} = C_{kj}$ always holds, we can apply the following conservative rule: the spectrum broker sets C_{jk} to 1 only when both users j and k confirm they have mutual interference. With this trick applied, colluding users cannot unilaterally fabricate an interference relationship to an innocent user who is honest, and they will lose their incentives to cheat because their efforts are in vain. If channels are asymmetric, however, the spectrum broker will have to ask secondary users to report their locations when there is a discrepancy between the reported C_{jk} and C_{kj}.

In sum, we examine secondary users' incentives to lie about the underlying interference relationships, and conclude no single user or group of users would have incentive to cheat individually or collusively, when the spectrum broker employ the conservative rule to determine the interference matrix $\mathbf{C}$ from secondary users' reports, under the condition of symmetric channels.

D. Complexity Issues

We have to examine the complexity of the proposed mechanism to see whether it is scalable when more users are involved in the auction game. Since the fully collusion-resistant pricing is a convex optimization problem when linear inequality constraints are known, they can be efficiently solved by numerical methods such as the interior point method [26]. However, one optimal allocation problem has to be solved to find the set of winners W, and another $2^{|W|} - 1$ problems have to be solved to obtain $U^*_{\mathbf{v}_{L(W\setminus W_C)}}$ used in the constraints. Unfortunately, the optimal allocation problem can be seen as the maximal weighted independent set problem [27] in graph theory, which is known to be NP-complete in general[5] even for the simplest case with $v_i = 1$ for all i [19]. As the computational complexity becomes formidable when the number of users N is large, the proposed auction mechanism seems unscalable. Therefore, near-optimal approximations with polynomial complexity are of great interest.

Lemma 1: Define $\boldsymbol{\mu}_{\mathbf{v}} = [\sqrt{v_1}, \sqrt{v_2}, \ldots, \sqrt{v_N}]^T$, the optimal allocation problem (4) with $\mathbf{x}^*$ as its optimizer is equivalent to the following optimization problem,

$$\begin{aligned} \widetilde{U}^*_{\mathbf{v}} = &\max_{\mathbf{y}} \ \left(\boldsymbol{\mu}_{\mathbf{v}}^T \mathbf{y}\right)^2, \\ \text{s.t.} \ & y_i y_j = 0, \ \forall i, j \text{ if } C_{ij} = 1, \\ & |\mathbf{y}|_2 = 1, \end{aligned} \tag{8}$$

whose optimizer $\mathbf{y}^*$ is given by $y_i^* = c\sqrt{v_i} x_i^*$ where c is a normalization constant such that $|\mathbf{y}^*|_2 = 1$.

Proof: See Appendix B. ■

The optimal allocation is no longer an integer programming problem, but still difficult to solve because of the non-convex feasible set [26]. To make it numerically solvable in polynomial time, the SDP relaxation can be applied, which enlarges the feasible set to a cone of positive semi-definite matrices (which is a convex set) by removing some constraints [20]. To this end, let $\mathbf{S} = \mathbf{y}\mathbf{y}^T$, i.e., $S_{ij} = y_i y_j$. The objective function in (8) becomes $\boldsymbol{\mu}_{\mathbf{v}}^T \mathbf{S} \boldsymbol{\mu}_{\mathbf{v}}$, and the two constraints turn out to be $S_{ij} = 0, \ \forall i, j$ if $C_{ij} = 1$ and $\text{tr}(\mathbf{S}) = 1$, respectively. The problem has to be optimized over $\{\mathbf{S} \in \mathscr{S}^N | \mathbf{S} = \mathbf{y}\mathbf{y}^T, \mathbf{y} \in \mathscr{M}^{N\times 1}\}$, or equivalently, $\{\mathbf{S} \in \mathscr{S}^N | \mathbf{S} \succeq \mathbf{O}, \text{rank}(\mathbf{S}) = 1\}$. Discarding the rank requirement while only keeping the positive semi-definite constraint, we arrive at the following convex optimization problem,

$$\begin{aligned} \vartheta(\mathbf{C}, \mathbf{v}) = &\max_{\mathbf{S}\succeq\mathbf{O}} \ \boldsymbol{\mu}_{\mathbf{v}}^T \mathbf{S} \boldsymbol{\mu}_{\mathbf{v}} \\ \text{s.t.} \ & \text{tr}(\mathbf{S}) = 1, \\ & S_{ij} = 0, \ \forall i, j \text{ if } C_{ij} = 1, \end{aligned} \tag{9}$$

which is also known as the *theta number* [28][29] in graph theory.

With the feasible set enlarged by relaxing a constraint to its necessary condition, the new optimization problem provides an upper bound to the original one: if the optimizer $\mathbf{S}^*$ can be decomposed as $\mathbf{S}^* = \mathbf{y}^* \mathbf{y}^{*T}$ which means $\mathbf{S}^*$ falls into the original feasible set, $\mathbf{y}^*$ will be the exact solution to (8); otherwise, $\vartheta(\mathbf{C}, \mathbf{v})$ is an upper bound that is unattainable. Fortunately, we verify by simulation that the near-optimal algorithm with relaxation performs well: in our problem setting, it gives the exact solution most of the time ($> 90\%$), and even for those unattainable cases, the bound is considerably tight since the average gap is within 5%.

As the upper bound is quite tight, we can replace $U^*_{\mathbf{v}_{L(W\setminus W_C)}}$ in (7) by its corresponding bound without significantly changing the prices charged to each winner; but we have to find out the optimizer $\mathbf{x}^*$ when deciding who are the winners. To this end, we examine whether $\mathbf{S}^*$ can be decomposed as the outer-product of a vector and itself, and if yes, we can get $\mathbf{y}^*$ and then map it back to $\mathbf{x}^*$. A much simpler way is to let $x_i^* = 1$ if $S_{ii}^* \neq 0$ since Lemma 1 tells us $x_i^* = 0$ if and only if $y_i^* = 0$, and then check whether this allocation is interference-free or not[6]. Most of the time, we will end up with the exact solution; and for those failed trials, we could resort to other suboptimal algorithms such as the greedy algorithm to find suboptimal allocation, or simply solve the original problem directly.

E. Physical Interference Model

In this subsection, we extend our auction mechanism to the situation where the physical model is employed to describe mutual interference. Now, the optimal allocation becomes social welfare maximization under physical interference constraints,

$$\begin{aligned} U^*_{\mathbf{v}} = &\max_{\mathbf{x}} \ \sum_{i=1}^{N} v_i x_i, \\ \text{s.t.} \ & \sum_{j=1}^{N} \alpha_{ji} x_j \leq 0, \ \text{if } x_i = 1, \\ & x_i = 0 \text{ or } 1, \ i = 1, 2, \ldots, N. \end{aligned} \tag{10}$$

Recall that those α's are defined as $\alpha_{ii} = -1$ and $\alpha_{ji} = \beta g_{ji}/g_{ii}, i \neq j$ in Section II, which basically depend on

[5] It is true except some special cases, e.g., when the graph is perfect. The graph corresponding to our optimal allocation problem does not possess those special properties.

[6] In practice, as $\mathbf{S}^*$ is solved by some numerical methods, the entries in $\mathbf{S}^*$ may not be strictly equal to 0. Thus, we can set $S_{ii}^* = 0$ whenever $|S_{ii}^*|$ is smaller than a threshold, say 10^{-5}.

channel gains. Thus, the optimal allocation remains much the same except that protocol interference constraints are replaced by physical interference constraints. Pricing strategies are similar, too.

Nevertheless, the SDP relaxation is a bit difficult because the constraints are much more complicated than constraints exerted by the protocol model. First, we replace the constraint "$\sum_{j=1}^{N}\alpha_{ji}x_j \le 0$ if $x_i = 1$" by an equivalent but compact form $x_i\left(\sum_{j=1}^{N}\alpha_{ji}x_j\right) \le 0$, because x_i is a binary integer variable. Then, we can apply similar approaches, i.e., $y_i = c\sqrt{v_i}x_i$ and $S_{ji} = y_j y_i$, and finally get the following relaxed optimization problem,

$$\begin{aligned} \max_{\mathbf{S}\succeq\mathbf{O}}\ & \boldsymbol{\mu}_{\mathbf{v}}^T\mathbf{S}\boldsymbol{\mu}_{\mathbf{v}} \\ \text{s.t. }\ & \text{tr}(\mathbf{S}) = 1, \\ & S_{ji} = 0,\ \forall i, j \text{ if } \alpha_{ji} > 1, \\ & \sum_{j=1}^{N}\frac{\alpha_{ji}}{\sqrt{v_j v_i}}S_{ji} \le 0,\ i = 1, 2, \ldots, N. \end{aligned} \tag{11}$$

Note that when $\alpha_{ji} > 1$, i.e., user j has strong interference on user i, user i cannot transmit simultaneously with user j, because if $x_i = x_j = 1$, we have $\sum_{j=1}^{N}\alpha_{ji}x_j \ge \alpha_{ji}x_j + \alpha_{ii}x_i = \alpha_{ji} - 1 > 0$ which will violate the constraint. Hence, the corresponding constraint is quite similar to that under the protocol model. Moreover, compared with (9), the SDP relaxation under the physical model (11) incorporates additional constraints reflecting the accumulation of interference power.

Additional difficulties rise when we recover the optimal allocation vector $\mathbf{x}^*$ from the optimizer $\mathbf{S}^*$. The reason is that under the protocol model we are able to map $\mathbf{y}^*$ back to $\mathbf{x}^*$ based on the converse part of Lemma 1, but it does not hold any more under the physical model. However, we can still exploit hints from $\mathbf{y}^*$ to construct a near-optimal $\widetilde{\mathbf{x}}$ using the greedy algorithm. Specifically, we sort $y_i^*/\sqrt{v_i}$ in a descending order, and denote the sorted index as $[n(1), n(2), \ldots, n(N)]$; for example, $n(1) = \arg\max_i \widetilde{y}_i/\sqrt{v_i}$. After initializing $\widetilde{\mathbf{x}}$ as an all-zero vector and setting $\widetilde{x}_{n(1)} = 1$, we set $\widetilde{x}_{n(2)} = 1$ if the resultant $\widetilde{\mathbf{x}}$ still satisfies the physical interference constraint and $\widetilde{x}_{n(2)} = 0$ otherwise. We determine binary values of $\widetilde{x}_{n(k)}, (k = 3, 4, \ldots, N)$ one by one in the same way, and finally obtain the vector $\widetilde{\mathbf{x}}$ which will be used as the allocation vector.

IV. Multi-Band Multi-Winner Auction

It is more interesting to study the case when M primary users want to lease their unused bands or a single primary user divides the band into M sub-bands for lease. In other words, there are M bands $(M > 1)$ available for secondary users to lease. In this section, we extend our existing results to the multi-band auction.

A. Multi-Band Auction Mechanism

Since usually there are a lot of secondary users competing for the spectrum resources, it is unfair if some users can access several bands while others are starved. In addition, if each secondary user is equipped with a single radio, the physical limitation will make it impossible to access several bands simultaneously. Therefore, we require each user should lease at most one band, and we further assume secondary users do not care which band they get, i.e., any band's value is v_i to user i.

Extending the one-band auction to a more general multi-band one, we have to find the counterpart of the auction mechanism including the optimal allocation and pricing strategies. As there are M sets of winners $W^1, W^2, \ldots, W^M$, we define M vectors $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^M$ correspondingly, where $x_i^m = 1$ indicates user i wins the mth band. Including the additional constraint that each user cannot lease more than one band, we have the M-band optimal allocation as follows,

$$\begin{aligned} U_{\mathbf{v}}^* = & \max_{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^M}\sum_{m=1}^{M}\sum_{i=1}^{N} v_i x_i^m, \\ & \text{s.t. }\ x_i^m + x_j^m \le 1,\ \forall i, j \text{ if } C_{ij} = 1, \forall m, \\ & \qquad \sum_{m=1}^{M} x_i^m \le 1,\ \forall i, \\ & \qquad x_i^m = 0 \text{ or } 1, i = 1, 2, \ldots, N; m = 1, 2, \ldots, M. \end{aligned} \tag{12}$$

In the multi-band auction, the set of losers becomes $L = \{1, 2, \ldots, N\} \setminus \bigcup_{j=1}^{M} W^j$ instead. Similar to the single-band partially collusion-resistant pricing strategy, the winners of the mth band have to pay the highest rejected bid from the losers, and the payment is split according to the NBS equilibrium,

$$\begin{aligned} & \max_{\{p_i\in[0,v_i],\, i\in W^m\}}\prod_{i\in W^m}(v_i - p_i), \\ & \qquad \text{s.t. }\sum_{i\in W^m} p_i = U_{\mathbf{v}_{-\left(\bigcup_{j=1}^{M} W^j\right)}}^*. \end{aligned} \tag{13}$$

The single-band fully collusion-resistant pricing strategy can be generalized too; for instance, the prices for the mth band are determined by

$$\begin{aligned} & \max_{\{p_i\in[0,v_i],\, i\in W^m\}}\prod_{i\in W^m}(v_i - p_i), \\ & \qquad \text{s.t. }\sum_{i\in W_C} p_i \ge U_{\mathbf{v}_{L(W^m\setminus W_C)}}^*, \forall W_C \subseteq W^m. \end{aligned} \tag{14}$$

When $M = 1$, the two pricing strategies reduce to the single-band case.

B. The SDP Relaxation for the Multi-Band Auction

In order to apply the SDP relaxation to the multi-band optimal allocation, we first introduce auxiliary variables $\boldsymbol{\chi} \triangleq \left[(\mathbf{x}^1)^T, (\mathbf{x}^2)^T, \ldots, (\mathbf{x}^M)^T\right]^T$ and $\boldsymbol{\nu} \triangleq \mathbf{1}_{M\times 1} \otimes \mathbf{v}$. Notice that for binary integers, the constraint $\sum_{m=1}^{M} x_i^m \le 1$ is equivalent to $x_i^m + x_i^k \le 1, \forall m \ne k$. Hence, the optimal allocation (12) can be written in an equivalent form,

$$\begin{aligned} \max_{\boldsymbol{\chi}}\ & \sum_{i=1}^{MN}\nu_i\chi_i, \\ \text{s.t. }\ & \chi_i + \chi_j \le 1,\ \forall i, j \text{ if } \Xi_{ij} = 1, \\ & \chi_i = 0 \text{ or } 1,\ i = 1, 2, \ldots, MN, \end{aligned} \tag{15}$$

where

$$\boldsymbol{\Xi} \triangleq \begin{bmatrix} \mathbf{C} & \mathbf{I} & \cdots & \mathbf{I} \\ \mathbf{I} & \mathbf{C} & \cdots & \mathbf{I} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{I} & \mathbf{I} & \cdots & \mathbf{C} \end{bmatrix} \tag{16}$$

is the effective interference matrix. Viewed as a single-band auction with MN users, the optimal allocation is upper bounded by the theta number,

$$\begin{aligned} \vartheta(\boldsymbol{\Xi}, \boldsymbol{\nu}) = &\max_{\mathbf{S} \succeq \mathbf{O}} \ \boldsymbol{\mu}_{\boldsymbol{\nu}}^T \mathbf{S} \boldsymbol{\mu}_{\boldsymbol{\nu}} \\ &\text{s.t.} \ \ \mathrm{tr}(\mathbf{S}) = 1, \\ &\qquad S_{ij} = 0, \ \forall i, j \text{ if } \Xi_{ij} = 1. \end{aligned} \tag{17}$$

This problem is optimized over $\mathbf{S} \in \mathscr{S}^{MN}$ which has $\frac{1}{2}MN(MN+1)$ degrees of freedom; however, there is some redundancy resulting from structural symmetry of the problem: according to our assumption that all bands are equivalent, interchanging the winners of band m and band k makes no difference. In general, if $\boldsymbol{\chi} = \left[\left(\mathbf{x}^1\right)^T, \left(\mathbf{x}^2\right)^T, \ldots, \left(\mathbf{x}^M\right)^T \right]^T$ is an optimal solution, for any permutation π, $\boldsymbol{\chi}(\pi) = \left[\left(\mathbf{x}^{\pi(1)}\right)^T, \left(\mathbf{x}^{\pi(2)}\right)^T, \ldots, \left(\mathbf{x}^{\pi(M)}\right)^T \right]^T$ will also be an optimal solution.

Similar to the one-to-one mapping in one-band optimal allocation proved by Lemma 1, we define $\boldsymbol{\eta} = c\left[\sqrt{\nu_1}\chi_1, \sqrt{\nu_2}\chi_2, \ldots, \sqrt{\nu_{MN}}\chi_{MN}\right] \triangleq \left[\left(\mathbf{y}^1\right)^T, \left(\mathbf{y}^2\right)^T, \ldots, \left(\mathbf{y}^M\right)^T \right]^T$ with c chosen such that $|\boldsymbol{\eta}|_2 = 1$. Due to symmetry, if $\mathbf{S} = \boldsymbol{\eta}\boldsymbol{\eta}^T$ is a solution to (17), so will be $\mathbf{S}(\pi) = \boldsymbol{\eta}(\pi)\boldsymbol{\eta}^T(\pi)$ where $\boldsymbol{\eta}(\pi) = \left[\left(\mathbf{y}^{\pi(1)}\right)^T, \left(\mathbf{y}^{\pi(2)}\right)^T, \ldots, \left(\mathbf{y}^{\pi(M)}\right)^T \right]^T$. The average of all $M!$ permutations is

$$\overline{\mathbf{S}} = \frac{1}{M!}\sum_{\pi} \mathbf{S}(\pi) = \frac{1}{M} \begin{bmatrix} \mathbf{S}_D & \mathbf{S}_F & \cdots & \mathbf{S}_F \\ \mathbf{S}_F & \mathbf{S}_D & \cdots & \mathbf{S}_F \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{S}_F & \mathbf{S}_F & \cdots & \mathbf{S}_D \end{bmatrix}, \tag{18}$$

where the diagonal blocks are

$$\mathbf{S}_D = \sum_{m=1}^{M} \left(\mathbf{y}^m\right)\left(\mathbf{y}^m\right)^T, \tag{19}$$

and the off-diagonal blocks are

$$\mathbf{S}_F = \frac{1}{M-1}\sum_{m=1}^{M}\sum_{k=1,k\neq m}^{M} \left(\mathbf{y}^m\right)\left(\mathbf{y}^k\right)^T. \tag{20}$$

As shown by the following proposition, we only have to optimize over two small matrices $\mathbf{S}_D$ and $\mathbf{S}_F$ rather than the large-dimension matrix $\mathbf{S}$ in (17). Just as the one-band case, the idea is to relax the specific structures of matrices $\mathbf{S}_D$ and $\mathbf{S}_F$ by necessary conditions that they should satisfy in terms of positive semi-definite properties.

Proposition 2: The multi-band optimal allocation (12) can be relaxed by the following optimization problem,

$$\begin{aligned} \max_{\mathbf{S}_D, \mathbf{S}_F} \quad & \boldsymbol{\mu}_{\mathbf{v}}^T(\mathbf{S}_D + (M-1)\mathbf{S}_F)\boldsymbol{\mu}_{\mathbf{v}} && (21) \\ \text{s.t.} \quad & \mathrm{tr}(\mathbf{S}_D) = 1, && (22) \\ & (S_D)_{ij} = 0, \ \forall i, j \text{ if } C_{ij} = 1, && (23) \\ & (S_F)_{ii} = 0, \ \forall i, && (24) \\ & \mathbf{S}_D \succeq \mathbf{O}, \ \mathbf{S}_D - \mathbf{S}_F \succeq \mathbf{O}, && (25) \\ & \mathbf{S}_D + (M-1)\mathbf{S}_F \succeq \mathbf{O}. && (26) \end{aligned}$$

Proof: The objective function is $\boldsymbol{\mu}_{\boldsymbol{\nu}}^T \overline{\mathbf{S}} \boldsymbol{\mu}_{\boldsymbol{\nu}}$. Since $\overline{\mathbf{S}} = \frac{1}{M}\left(\mathbf{I}_{M\times M} \otimes \mathbf{S}_D + (\mathbf{1}_{M\times M} - \mathbf{I}_{M\times M}) \otimes \mathbf{S}_F\right)$ and $\boldsymbol{\mu}_{\boldsymbol{\nu}} = \mathbf{1}_{M\times 1} \otimes \boldsymbol{\mu}_{\mathbf{v}}$, we apply the properties of Kronecker products [30],

$$\begin{aligned} \boldsymbol{\mu}_{\boldsymbol{\nu}}^T \overline{\mathbf{S}} \boldsymbol{\mu}_{\boldsymbol{\nu}} = &\frac{1}{M}((\mathbf{1}_{M\times 1}^T \mathbf{I}_{M\times M} \mathbf{1}_{M\times 1}) \otimes (\boldsymbol{\mu}_{\mathbf{v}}^T \mathbf{S}_D \boldsymbol{\mu}_{\mathbf{v}}) + \\ &(\mathbf{1}_{M\times 1}^T(\mathbf{1}_{M\times M} - \mathbf{I}_{M\times M})\mathbf{1}_{M\times 1}) \otimes (\boldsymbol{\mu}_{\mathbf{v}}^T \mathbf{S}_F \boldsymbol{\mu}_{\mathbf{v}})) \\ = &\boldsymbol{\mu}_{\mathbf{v}}^T(\mathbf{S}_D + (M-1)\mathbf{S}_F)\boldsymbol{\mu}_{\mathbf{v}}. \end{aligned} \tag{27}$$

Because $|\boldsymbol{\eta}|_2 = 1$, we have $1 = \boldsymbol{\eta}^T\boldsymbol{\eta} = \sum_{m=1}^{M} \left(\mathbf{y}^m\right)^T \left(\mathbf{y}^m\right) = \sum_{m=1}^{M} \mathrm{tr}\left(\left(\mathbf{y}^m\right)\left(\mathbf{y}^m\right)^T\right) = \mathrm{tr}(\mathbf{S}_D)$, and hence we obtain constraint (22).

Moreover, the interference constraints in the original problem (12) imply that $y_i^m y_j^m = 0$ if $C_{ij} = 1$ and $y_i^m y_i^k = 0$ $\forall m \neq k$, according to the relationship of $\mathbf{x}$ and $\mathbf{y}$ established in Lemma 1. Hence, $\forall i, j$ such that $C_{ij} = 1$, $(S_D)_{ij} = \sum_{m=1}^{M} y_i^m y_j^m = 0$, which is the constraint (23). Constraint (24) can be proved in a similar way.

The final step is the SDP relaxation. To show a matrix $\mathbf{S} \in \mathscr{S}^N$ is positive semi-definite, we prove by definition that $\mathbf{z}^T\mathbf{S}\mathbf{z} \geq 0$ for any vector $\mathbf{z} \in \mathscr{M}^{N\times 1}$. Given any vector $\mathbf{z}$, define scalars $z^m = \mathbf{z}^T\mathbf{y}^m, m = 1, 2, \ldots, M$. Then,

$$\mathbf{z}^T\mathbf{S}_D\mathbf{z} = \sum_{m=1}^{M} (z^m)^2 \geq 0. \tag{28}$$

In addition,

$$\begin{aligned} \mathbf{z}^T(\mathbf{S}_D - \mathbf{S}_F)\mathbf{z} &= \sum_{m=1}^{M}(z^m)^2 - \frac{1}{M-1}\sum_{m=1}^{M}\sum_{k=1,k\neq m}^{M} z^m z^k \\ &= \frac{1}{M-1}\left(M\sum_{m=1}^{M}(z^m)^2 - \sum_{m=1}^{M}\sum_{k=1}^{M} z^m z^k\right) \\ &= \frac{1}{M-1}\left(M\sum_{m=1}^{M}(z^m)^2 - \left(\sum_{m=1}^{M} z^m\right)^2\right) \geq 0, \end{aligned} \tag{29}$$

where $M\sum_{m=1}^{M}(z^m)^2 \geq \left(\sum_{m=1}^{M} z^m\right)^2$ follows the Cauchy-Schwartz inequality. Also,

$$\begin{aligned} \mathbf{z}^T(\mathbf{S}_D + (M-1)\mathbf{S}_F)\mathbf{z} &= \sum_{m=1}^{M}(z^m)^2 + \sum_{m=1}^{M}\sum_{k=1,k\neq m}^{M} z^m z^k \\ &= \sum_{m=1}^{M}\sum_{k=1}^{M} z^m z^k = \left(\sum_{m=1}^{M} z^m\right)^2 \geq 0. \end{aligned} \tag{30}$$

Therefore, matrices $\mathbf{S}_D$, $\mathbf{S}_D - \mathbf{S}_F$, and $\mathbf{S}_D + (M-1)\mathbf{S}_F$ are all positive semi-definite. ■

We verify by simulation that this upper bound is also tight and hence can be employed in our auction to reduce complexity. As the new optimization problem is optimized over two symmetric matrices $\mathbf{S}_D, \mathbf{S}_F \in \mathscr{S}^N$, the total number of degrees of freedom is $N(N+1)$, which is significantly smaller than that of direct relaxation $\frac{1}{2}MN(MN+1)$. Roughly speaking, degrees of freedom, as an important factor affecting the computational complexity, are reduced from $O(M^2N^2)$ to $O(N^2)$.

V. Simulation Results

In this section, we evaluate the performance of the proposed collusion-resistant multi-winner spectrum auction mechanisms by computer experiments. Consider a 1000×1000 m^2 area, in which N secondary users are uniformly distributed. Assume each secondary user is a cognitive base station with R_I-meter coverage radius, and according to the protocol model, two users at least $2R_I$ meters away can share the same band without mutual interference. We use two values for R_I: $R_I = 150$ for a light-interference network, and $R_I = 350$ for a heavy-interference network. The valuations of different users $\{v_1, v_2, \ldots, v_N\}$ are assumed to be i.i.d. random variables uniformly distributed in $[20, 30]$.

We consider the one-band auction, i.e., $M = 1$. Fig. 2 shows the seller's revenue versus the number of secondary users when different auction mechanisms are employed. The result is averaged over 100 independent runs, in which the locations and valuations of the N secondary users are generated randomly with uniform distribution. As shown in the figure, directly applying the second-price scheme under-utilizes spectrum resources, and the VCG mechanism also suffers from low revenue. The proposed collusion-resistant methods, however, significantly improve the primary user's revenue, e.g., nearly 15% increase compared to the VCG outcome when $R_I = 350$, and 30% increase when $R_I = 150$. This means the proposed algorithms have better performance when more secondary users are admitted to lease the band simultaneously.

Moreover, the proposed auction mechanisms can effectively combat user collusion. We use the percentage of the system utility taken away by colluders to represent the vulnerability to sublease colluding attacks. Fig. 3 demonstrates the results from 100 independent runs. For example, when $R_I = 150$ and there are 20% colluders, colluders may steal away up to 10% of the system utility with the VCG pricing mechanism, and much more profits could be taken away by colluders if more secondary users become colluders. To protect the primary user's benefit, collusion-resistant mechanisms can be applied. As show in the figure, the partially collusion-resistant pricing strategy may be not as good as the VCG mechanism on average under some circumstances because it cannot completely remove sublease collusion, but it makes the worst-case colluding gains drop considerably; for instance, when $R_I = 150$ and all users are able to collude, more than half of the system utility could be taken away if the VCG pricing is used, but only 22% with the partially collusion-resistant pricing method. The fully collusion-resistant pricing strategy, as expected, completely eliminates collusion, and hence is an ideal choice when there is a risk of sublease collusion.

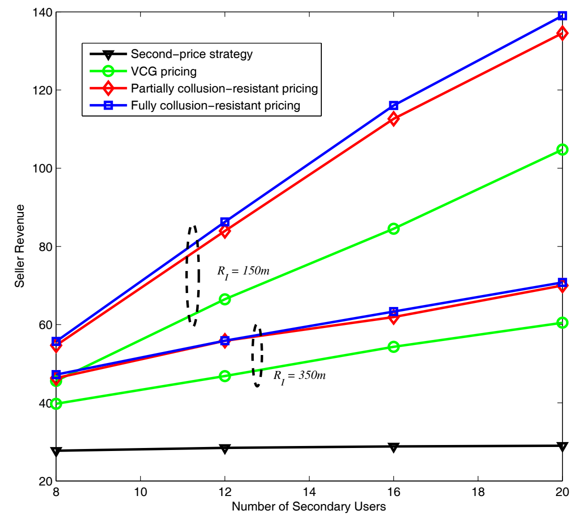

Fig. 2. Seller's revenue when different auction mechanisms are employed.

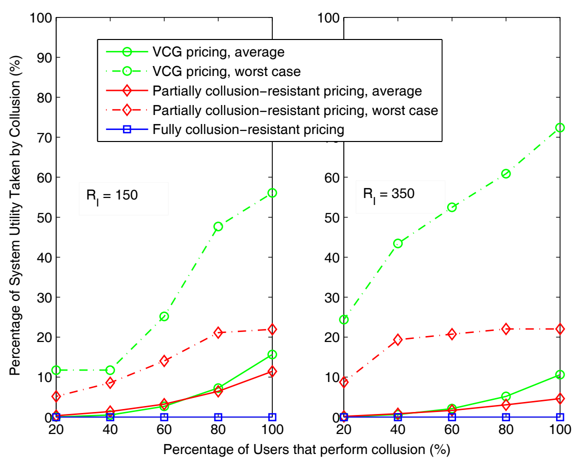

Fig. 3. Normalized collusion gains under different auction mechanisms versus the percentage of colluders in a spectrum auction with $N = 20$ secondary users.

The performance of the near-optimal algorithm is presented in Fig. 4. As shown by the simulation results, the near-optimal algorithm can yield the exact solution in more than 90% of the total runs. Even for those that the near-optimal algorithm fails to return the exact solution, it can still yield a tight upper bound with the average difference less than 5%; to show the robustness of the algorithm, we further provide the 90% confidence intervals (i.e., the range that 90% of the data fall in), which show that the gap between the near-optimal solution and the exact solution is within 10%.

Finally, we show the reduction of complexity in terms of the processing time when optimization is done in MATLAB[7]. In

[7]To solve the SDP problem, CVX software [31], a MATLAB-based software for disciplined convex optimization, is employed.

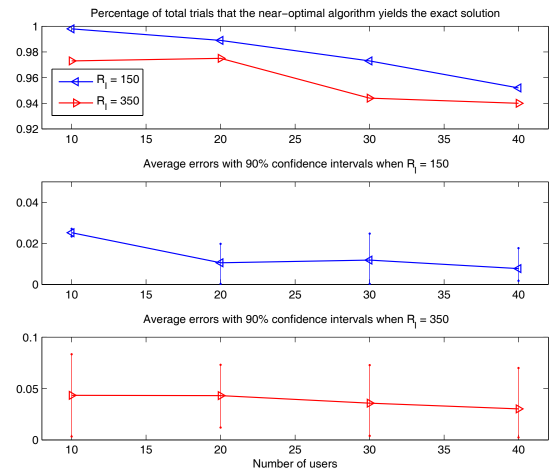

Fig. 4. The percentage of total trials that the near-optimal algorithm yields the exact solution (upper), and the average difference with 90% confidence intervals (i.e., from 5% to 95%) between the near-optimal solution and the exact solution for those failed trials (middle and lower, for $R_I = 150$ and 350, respectively).

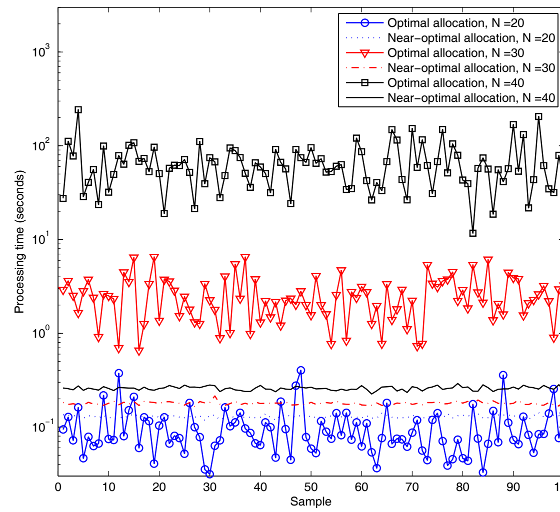

Fig. 5. Sampled processing time of the optimal allocation and the near-optimal allocation with the SDP relaxation, when the number of users $N = 20, 30, 40$, respectively.

Fig. 5, the processing time of solving the optimal allocation problem is compared with that of solving the near-optimal allocation problem in 100 independent runs, when $R_I = 350$ and the number of users is N =20, 30, and 40, respectively. With N increasing, the time to find the optimal solution increases dramatically, whereas the time to find an near-optimal solution using the SDP relaxation only increases slightly. Moreover, the processing time of the optimal algorithm fluctuates considerably in different realizations, but the processing time with the SDP relaxation shows small variation. Next, we show for multi-band allocation, applying relaxation with reduced dimension (given in Proposition 2) can further reduce the complexity of the straightforward relaxation. We fix $N = 40$, increase the number of bands M from 2 up to 6, and present the average of processing time in 100 independent runs in Fig. 6. As shown in the figure, the complexity results in terms of processing time are consistent with the analysis in the previous section, that is, solving the near-optimal allocation using reduced-dimension relaxation is more efficient when M is large.

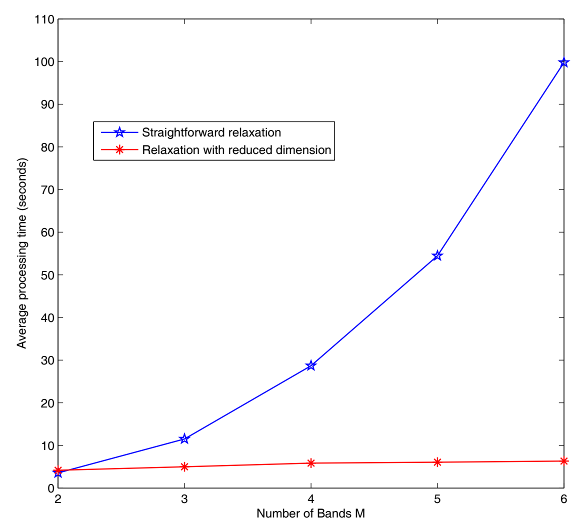

Fig. 6. Average processing time of the straightforward SDP relaxation and the SDP relaxation with reduced dimension, when the number of bands M increases.

VI. CONCLUSIONS

In this paper, we present a novel multi-winner auction game for the spectrum auction scenario in cognitive radio networks, in which secondary users can lease some temporarily unused bands from primary users. As this kind of auction does not exist in the literature where commodities are usually quantity-limited, suitable auction mechanisms are developed to guarantee full efficiency of the spectrum utilization, yield higher revenue to primary users, and help eliminate user collusion. To make the proposed scheme scalable, the SDP relaxation is applied to get a near-optimal solution in polynomial time. Moreover, we extend the one-band auction mechanism to the multi-band case. Simulation results are presented to demonstrate performance and complexity of proposed auction mechanisms.

APPENDIX A
PROOF OF PROPOSITION 1

Proof: Let $q_i = v_i - p_i$ for $i \in W$ and use the fact $\sum_{i\in W} v_i = U^*_{\mathbf{v}}$, the optimization problem (5) is equivalent to the following convex optimization problem

$$\begin{aligned} \min_{\{q_i\in[0,v_i],\, i\in W\}} \; & -\sum_{i\in W} \ln q_i, \\ \text{s.t.} \; & \sum_{i\in W} q_i = U^*_{\mathbf{v}} - U^*_{\mathbf{v}_{-W}} \triangleq \Delta U. \end{aligned} \tag{A.31}$$

After introducing the Lagrange multipliers λ and $\mu_i \geq 0, i \in W$, the Lagrangian function is

$$L(\mathbf{q}, \lambda, \boldsymbol{\mu}) = -\sum_{i\in W} \ln q_i + \lambda(\sum_{i\in W} q_i - \Delta U) + \sum_{i\in W} \mu_i(q_i - v_i), \tag{A.32}$$

from which Karush-Kuhn-Tucker (KTT) conditions [26] can be obtained as follows,

$$q_i = \frac{1}{\lambda + \mu_i},\ \mu_i \geq 0,\ \mu_i(q_i - v_i) = 0,\ \sum_{i\in W} q_i = \Delta U. \tag{A.33}$$

Define $\rho = \frac{1}{\lambda}$. For those i's such that $q_i = v_i$, $q_i = \frac{1}{\lambda+\mu_i} \leq \frac{1}{\lambda} = \rho$; for other i's such that $q_i < v_i$, the third condition implies $\mu_i = 0$, and thus $q_i = \frac{1}{\lambda+\mu_i} = \rho$. Therefore, the solution is

$$q_i = \min(v_i, \rho), \tag{A.34}$$

with ρ chosen such that the last condition in (A.33) is satisfied. Finally, $p_i = v_i - q_i$ yields (6). In particular, if $\frac{\Delta U}{|W|} \leq \min_i(v_i)$, $\rho = \frac{\Delta U}{|W|}$ and $p_i = v_i - \rho$ will be the solution. ■

APPENDIX B
PROOF OF LEMMA 1

Proof: Assume W^* and V^* are the support of the optimizers $\mathbf{x}^*$ and $\mathbf{y}^*$, respectively, i.e., $i \in W^*$ if and only if $x_i^* \neq 0$, and $i \in V^*$ if and only if $y_i^* \neq 0$.

Note that the constraint $x_i + x_j \leq 1$ can also be written as $x_i x_j = 0$ for binary integers. Define $y_i = \frac{\sqrt{v_i} x_i^*}{\sqrt{\sum_{k\in W^*} v_k}}$ whose norm $|\mathbf{y}|_2$ equals 1, and moreover, for i, j such that $C_{ij} = 1$, $y_i y_j = \frac{\sqrt{v_i v_j}}{\sum_{k\in W^*} v_k} x_i^* x_j^* = 0$. Satisfying both constraints, $\mathbf{y}$ is in the feasible set, which should yield a value not exceeding the optimum,

$$\widetilde{U}_{\mathbf{v}}^* \geq \left(\boldsymbol{\mu}_{\mathbf{v}}^T \mathbf{y}\right)^2 = \left(\frac{\sum_{i\in W^*} v_i}{\sqrt{\sum_{k\in W^*} v_k}}\right)^2 = \sum_{i\in W^*} v_i = U_{\mathbf{v}}^*. \tag{B.35}$$

On the other hand, knowing $\mathbf{y}^*$ is the optimizer, we can confine the problem to V^* as follows,

$$\max_{y_i, i\in V^*} \left(\sum_{i\in V^*} \sqrt{v_i} y_i\right)^2, \quad \text{s.t.} \sum_{i\in V^*} y_i^2 = 1. \tag{B.36}$$

According to the Cauchy-Schwartz inequality, $\left(\sum_{i\in V^*} \sqrt{v_i} y_i\right)^2 \leq \left(\sum_{i\in V^*} v_i\right)\left(\sum_{i\in V^*} y_i^2\right) = \sum_{i\in V^*} v_i$, where the equality holds when $y_i^* = c\sqrt{v_i}\,(i \in V^*)$ for some constant c. Furthermore, it is impossible to find $i, j \in V^*$ such that $C_{ij} = 1$; otherwise, $y_i^* y_j^* \neq 0$ will violate the constraint. This implies V^* is also a compatible group of users without interference, and we have

$$\widetilde{U}_{\mathbf{v}}^* = \sum_{i\in V^*} v_i \leq U_{\mathbf{v}}^*. \tag{B.37}$$

Comparing (B.35) with (B.37), we conclude that $\widetilde{U}_{\mathbf{v}}^* = U_{\mathbf{v}}^*$, and the optimizers are related by $y_i^* = c\sqrt{v_i} x_i^*$ with the normalization factor c. ■

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