An example of a signaling game

- **Two period reputation game**
  
  - 2 firms on the market
  
  **Period 1:** both firms of the market; only firm 1 action \(a_1\)
  
  - Actions for firm 1: prey or accommodate
    
    - If prey: firm 2 gets \(P_2<0\)
    
    - If accommodate: firm 2 gets \(D_2>0\)
  
  - Type of firm 1: “sane” or “crazy”
    
    - “crazy” – always prey
    
    - “sane”
      
      - If accommodates: payoff for 1: \(D_1>0\)
      
      - If preys: payoff for 1: \(P_1< D_1 \rightarrow\) prefers to accommodate
      
      - If 2 exits \(\rightarrow\) \(M_1>D_1\) (monopoly)
Two period reputation game: cont.

- **Period 2** - player 2 selects action $a_2$: stay or exit
  - If exits, it gets a 0 payoff, and player 1 gets $M_1 > D_1$
- **Assumptions:**
  - Player 1 knows his type
  - Player 2 believes that player 1 is sane with probability $p$
  - $\delta$ = discount factor between the two periods
- **Building reputation for the sane player**
  - Player 1 may try to convince player 2 that he is crazy, to get $M_1 > D_1$ in the second period of the game.
Taxonomy of PBE

- **Separating equilibrium:** the two types of player 1, choose two different actions in period 1
  - Firm 2 has complete information for the second period
    \[ \mu(\theta = \text{sane} \mid a_1 = \text{accommodate}) = 1 \]
    \[ \mu(\theta = \text{crazy} \mid a_1 = \text{prey}) = 1 \]

- **Pooling equilibrium:** the two types of player 1, choose the same action in period 1
  \[ \mu(\theta = \text{sane} \mid a_1 = \text{prey}) = p \]

- **Hybrid (semi-separating equilibria):** the same type may randomize between preying and accommodating
  \[ \mu(\theta = \text{sane} \mid a_1 = \text{prey}) \in (0, p) \]
  \[ \mu(\theta = \text{sane} \mid a_1 = \text{accommodate}) = 1 \]

- Separating eq. existence: sufficient and necessary condition

\[ D_1 (1 + \delta) \geq P_1 + \delta M_1 \iff \delta (M_1 - D_1) \leq (D_1 - P_1) \]

- Pooling eq. \( \Rightarrow \) to enforce exit for player 2. Condition:

\[ pD_2 + (1 - p)P_2 \leq 0 \]

- If the above two conditions do not hold \( \Rightarrow \) hybrid PBE

Note: uniqueness of the eq. in this case, is due to the fact that the “crazy” type is assumed to always prey.
Multi-stage games with observed actions and incomplete information

- Each player \( i \) has type \( \theta_i \), and types are independent
  \[
p(\theta) = \prod_{i=1}^{I} p_i(\theta_i)
\]
- At each period \( t \) (\( t=0,1,2,\ldots, T \)), players choose their actions simultaneously, and the actions are revealed at the end of the period.
- Players’ action set at a date \( t \) is type independent
- Behavior strategy: \( \sigma_i(a_i \mid h^t, \theta) \)
- Payoffs \( u_i(h^{T+1}, \theta) \)

Subgame perfection \( \Rightarrow \) BNE not only for the whole game, but also for the “continuation game” starting at period \( t \) after all possible histories \( h^t \)
  - Continuation games \( \Rightarrow \) proper subgames?
    - No. They do not stem from a singleton information set
Continuation games $\rightarrow$ true games

- Need to specify the players’ beliefs at the start of each continuation game.

**Definition:** A perfect Bayesian equilibrium is a $(\sigma, \mu)$ that satisfies (P) and (B(i) – B(iv)).

B(i) Posterior beliefs are independent, and all types of player i have the same belief.

- For all $\theta$, $t$ and $h^t$:

$$\mu_i(\theta_{-i} | \theta_i, h^t) = \prod_{j \neq i} \mu_i(\theta_j | h^t)$$

- even unexpected events will not change the independence assumption for the type of the opponents
Perfect Bayesian equilibrium: cont

- B(ii) Beliefs are updated according to Bayes’ rule:
  - For all $i,j$, $h^t$, and $a^t_j$, If there exist
  \[
  \hat{\theta}_j, \text{ s.t. } \mu_i(\theta_j | h^t) > 0, \quad \sigma_j(a^t_j | h^t, \theta_j) > 0, \text{ then}
  \]
  \[
  \mu_i(\theta_j | h^t, a^t_j) = \frac{\mu_i(\theta_j | h^t)\sigma_j(a^t_j | h^t, \theta_j)}{\sum_{\hat{\theta}_j} \mu_i(\hat{\theta}_j | h^t)\sigma_j(a^t_j | h^t, \hat{\theta}_j)}
  \]

- B(iii) Don’t signal what you don’t know
  - For all $i,j$, $h^t$, and $a^t$ and $\hat{a}^t$
  \[
  \mu_i(\theta_j | (h^t, a^t)) = \mu_i(\theta_j | (h^t, \hat{a}^t)), \text{ if } a^t_j = \hat{a}^t_j
  \]
Perfect Bayesian eq. – cont.

• B(iv) All players have to have the same belief about the type of another player
  – Imposed because of the req. of eq. analysis: players have the same belief about each other’s strategies.
  – For all $\theta_k$, and $h^t$

$$
\mu_i(\theta_k | h^t) = \mu_k(\theta_k | h^t) = \mu(\theta_k | h^t), \text{ for } i \neq j \neq k
$$

• (P) For each player $i$, type $\theta_i$, alternative strategy $\sigma'_i$, and history $h^t$

$$
u_i(\sigma | h^t, \theta_i, \mu(.) | h^t) \geq u_i(\sigma'_i, \sigma_{-i} | h^t, \theta_i, \mu(.) | h^t)$$
Sequential equilibrium

- We saw already that the requirement that the players’ strategies form a Nash equilibrium is too weak → formally the only proper subgame for the games of incomplete, or imperfect information is the whole game.
- Recall the initial example

  ![Game Diagram]

  - Nash equilibria: (L,A) and (R,B)
  - Both subgame perfect

Is (LA) equilibrium plausible?
Sequential equilibrium: cont.

• \((L,A)\) is not plausible
  - Whatever player's 2 beliefs on player's 1 move (M or R), he must chose B if he has an opportunity to move.

• Need to generalize the previous condition \((P) \rightarrow \) given the system of beliefs, no player can gain by deviating at any information set.

• (s) An assessment \((\sigma,\mu)\) is *sequentially rational* if, for any information set \(h\), and alternative strategy \(\sigma'_{i(h)}\),

\[
  u_{i(h)}(\sigma | h, \mu(h)) \geq u_{i(h)}(\sigma'_{i(h)}, \sigma_{-i(h)} | h, \mu(h))
\]
Sequential eq. cont.

- Consistency condition on beliefs is also introduced.

- (C) An assessment \((\sigma, \mu)\) is consistent if

\[
(\sigma, \mu) = \lim_{n \to +\infty} (\sigma^n, \mu^n)
\]

For some sequence \((\sigma^n, \mu^n) \in \Psi^0\) → The set of all assessments.

- **Definition:** A sequential equilibrium is an assessment \((\sigma, \mu)\) that satisfies (S) and (C).

**Existence:** For any finite extensive-form game, there exist at least one sequential equilibrium.