Perfect Bayesian Equilibrium for Multi-stage Games

• **The Basic Signaling Game** (2 players)

  • Player 1: leader (sender)
  • Player 2: follower (receiver)
  • Player 1 has private info. about its type $\theta \in \Theta$, action $a_1 \in A_1$
  • Player 2: its type is common knowledge, action $a_2 \in A_2$
  • Space of mixed actions: $A_1$ and $A_2$, with elements $\alpha_1$ and $\alpha_2$
  • Utility of player i: $u_i(\alpha_1, \alpha_2, \theta)$
  • Player 2 – prior belief about player’s 1 type: $p \rightarrow$ common knowledge
  • Strategy for player 1: probability distribution $\sigma_1(\cdot | \theta)$ over actions $a_1 \in A_1$ for each type $\theta$
  • Strategy for player 2: probability distribution $\sigma_2(\cdot | \theta)$ over actions $a_2 \in A_2$ for each $a_1 \in A_1$
Payoffs Calculation

- Payoff for player 1, given its type $\theta$:

$$u_1(\sigma_1, \sigma_2, \theta) = \sum_{a_1} \sum_{a_2} \sigma_1(a_1 | \theta) \sigma_2(a_2 | a_1) u_1(a_1, a_2, \theta)$$

- Player 2’s (ex ante – beforehand payoff) to strategy $\sigma_2(. | a_1)$, when player 1 plays $\sigma_1(. | \theta)$:

$$\sum_{\theta} p(\theta) \left[ \sum_{a_1} \sum_{a_2} \sigma_1(a_1 | \theta) \sigma_2(a_2 | a_1) u_2(a_1, a_2, \theta) \right]$$

- Should we make the decision based on the above computed payoff?
Posterior beliefs

- Player 2, observes the action of player 2 \( \Rightarrow \) must update its belief on \( \theta \), and its choice of action \( \Rightarrow \) posterior distribution over \( \Theta \): \( \mu(.|a_1) \)

- How to compute \( \mu(.|a_1) \)?
  - Player 1 actions may depend on its type
  - Let \( \sigma_1^*(.|\theta) \) to be player 1 strategy
  - Know \( p(.) \), \( \sigma_1^*(.|\theta) \) and observe \( a_1 \): use Bayes rule

- Extension of subgame-perfect eq. \( \Rightarrow \) perfect Bayesian eq.
  - Player 2 max. its payoff conditional on \( a_1 \):

\[
\sum_{\theta} \mu(\theta | a_1) u_2(a_1, \sigma_2(. | a_1), \theta) = \sum_{\theta} \sum_{a_2} \mu(\theta | a_1) \sigma_2(a_2 | a_1) u_2(a_1, a_2, \theta)
\]
Perfect Bayesian Equilibrium (PBE)

- **Definition**: A perfect Bayesian eq. of a signaling game is a strategy profile \( \sigma^* \) and posterior beliefs \( \mu(.|a_1) \), s.t.:

  - (P1): \( \forall \theta, \; \sigma^*_1(\cdot | \theta) \in \arg \max_{\alpha_1} u_1(\alpha_1, \sigma^*_2, \theta) \)

  - (P2): \( \forall a_1, \; \sigma^*_2(\cdot | a_1) \in \arg \max_{\alpha_2} \sum_{\theta} \mu(\theta | a_1) u_2(a_1, \alpha_2, \theta) \)

- (B)

  \[
  \mu(\theta | a_1) = \frac{p(\theta) \sigma^*_1(a_1 | \theta)}{\sum_{\theta' \in \Theta} p(\theta') \sigma^*_1(a_1 | \theta')}, \quad \text{if} \quad \sum_{\theta' \in \Theta} p(\theta') \sigma^*_1(a_1 | \theta') > 0
  \]

  \[
  \mu(.|a1), \text{ is any prob. distr. on } \Theta, \text{ if } \sum_{\theta' \in \Theta} p(\theta') \sigma^*_1(a_1 | \theta') = 0
  \]
Backward induction solution?

- Backward induction was used in games with perfect information.
- PBE: strategies are optimal given the beliefs and the beliefs are obtained from equilibrium strategies and observed actions.
  - Circularity $\rightarrow$ PBE cannot be determined by backward induction.
An example of a signaling game

- **Two period reputation game**
  - 2 firms on the market
  - **Period 1**: both firms of the market; only firm 1 action $a_1$
  - Actions for firm 1: prey or accommodate
    - If prey: firm 2 gets $P_2<0$
    - If accommodate: firm 2 gets $D_2>0$
  - Type of firm 1: “sane” or “crazy”
    - “crazy” – always prey
    - “sane”
      - If accommodates: payoff for 1: $D_1>0$
      - If preys: payoff for 1: $P_1 < D_1$ → prefers to accommodate
      - If 2 exits $M_1 > D_1$ (monopoly)
Two period reputation game: cont.

- Period 2 - player 2 selects action a2: stay or exit
  - If exits, it gets a 0 payoff, and player 1 gets M1 > D1

- Assumptions:
  - Player 1 knows his type
  - Player 2 believes that player 1 is sane with probability p
  - $\delta$ = discount factor between the two periods

- Building reputation for the sane player
  - Player 1 may try to convince player 2 that he is crazy, to get M1 > D1 in the second period of the game.
Taxonomy of PBE

- **Separating equilibrium:** the two types of player 1, choose two different actions in period 1
  - Firm 2 has complete information for the second period
    \[
    \mu(\theta = \text{sane} \mid a_1 = \text{accomodate}) = 1 \\
    \mu(\theta = \text{crazy} \mid a_1 = \text{prey}) = 1
    \]

- **Pooling equilibrium:** the two types of player 1, choose the same action in period 1
  \[
  \mu(\theta = \text{sane} \mid a_1 = \text{prey}) = p
  \]

- **Hybrid (semi-separating equilibria):** the same type may randomize between preying and accommodating
  \[
  \mu(\theta = \text{sane} \mid a_1 = \text{prey}) \in (0, p) \\
  \mu(\theta = \text{sane} \mid a_1 = \text{accommodate}) = 1
  \]

- Separating eq. existence: sufficient and necessary condition
  \[ D_1(1 + \delta) \geq P_1 + \delta M_1 \iff \delta(M_1 - D_1) \leq (D_1 - P_1) \]

- Pooling eq. \( \rightarrow \) to enforce exit for player 2. Condition:
  \[ pD_2 + (1 - p)P_2 \leq 0 \]

- If the above two conditions do not hold \( \rightarrow \) hybrid PBE

Note: uniqueness of the eq. in this case, is due to the fact that the “crazy” type is assumed to always prey.
Multi-stage games with observed actions and incomplete information

- Each player $i$ has type $\theta_i$, and types are independent
  $$p(\theta) = \prod_{i=1}^{I} p_i(\theta_i)$$

- At each period $t$ ($t=0,1,2,\ldots,T$), players choose their actions simultaneously, and the actions are revealed at the end of the period.

- Players’ action set at a date $t$ is type independent

- Behavior strategy: $\sigma_i(a_i \mid h^t, \theta)$

- Payoffs $u_i(h^{T+1}, \theta)$

- Subgame perfection $\Rightarrow$ BNE not only for the whole game, but also for the “continuation game” starting at period $t$ after all possible histories $h^t$
  - Continuation games $\Rightarrow$ proper subgames?
    - No. They do not stem from a singleton information set
Continuation games $\rightarrow$ true games

- Need to specify the players’ beliefs at the start of each continuation game.
- **Definition:** A perfect Bayesian equilibrium is a $(\sigma, \mu)$ that satisfies (P) and (B(i) – B(iv)).

B(i) Posterior beliefs are independent, and all types of player $i$ have the same belief.
- For all $\theta$, $t$ and $h^t$:
  \[
  \mu_i(\theta_{-i} \mid \theta_i, h^t) = \prod_{j \neq i} \mu_i(\theta_j \mid h^t)
  \]
- even unexpected events will not change the independence assumption for the type of the opponents