Game Theory

Department of Electronics

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Lecture 11

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Perfect Bayesian Equilibrium for Multi-stage Games

- **The Basic Signaling Game (**2 players)
  - Player 1: leader (sender)
  - Player 2: follower (receiver)
  - Player 1 has private inf. about its type $\theta \in \Theta$, action $a_1 \in A_1$
  - Player 2: its type is common knowledge, action $a_2 \in A_2$
  - Space of mixed actions: $\mathcal{A}_1$ and $\mathcal{A}_2$, with elements α_1 and α_2
  - Utility of player i: $u_i(\alpha_1, \alpha_2, \theta)$
  - Player 2 – prior belief about player's 1 type: *p* → common knowledge
  - Strategy for player 1: probability distribution $\sigma_1(.| \theta)$ over actions $a_1 \in A_1$ for each type θ
  - Strategy for player 2: probability distribution $\sigma_2(.| \theta)$ over actions $a_2 \in A_2$ for each $a_1 \in A_1$

Payoffs Calculation

- Payoff for player 1, given its type θ:

$$u_1(\sigma_1, \sigma_2, \theta) = \sum_{a_1} \sum_{a_2} \sigma_1(a_1 \mid \theta) \sigma_2(a_2 \mid a_1) u_1(a_1, a_2, \theta)$$

- Player 2's (ex ante – beforehand payoff) to strategy $\sigma_2(.|a_1)$, when player 1 plays $\sigma_1(.|\theta)$:

$$\sum_{\theta} p(\theta) \left[\sum_{a_1} \sum_{a_2} \sigma_1(a_1 \mid \theta) \sigma_2(a_2 \mid a_1) u_2(a_1, a_2, \theta) \right]$$

- Should we make the decision based on the above computed payoff?

Posterior beliefs

- Player 2, observes the action of player 2 → must update its belief on θ, and its choice of action → posterior distribution over Θ: $\mu(.|a_1)$
- How to compute $\mu(.|a_1)$?
  - Player 1 actions may depend on its type
  - Let $\sigma_1^*(.|\ \theta)$ to be player 1 strategy
  - Know p(.), $\sigma_1^*(.|\ \theta)$ and observe a_1: use Bayes rule
- Extension of subgame-perfect eq. → perfect Bayesian eq.
  - Player 2 max. its payoff conditional on a_1:

$$\sum_{\theta} \mu(\theta \mid a_1) u_2(a_1, \sigma_2(. \mid a_1), \theta) = \sum_{\theta} \sum_{a_2} \mu(\theta \mid a_1) \sigma_2(a_2 \mid a_1) u_2(a_1, a_2, \theta)$$

Perfect Bayesian Equilibrium (PBE)

- **Definition:** A perfect Bayesian eq. of a signaling game is a strategy profile σ^* and posterior beliefs $\mu(.|a_1)$, s.t.:

- (P1): $\forall \theta,\ \sigma_1^*(.|\theta) \in \arg\max_{\alpha_1} u_1(\alpha_1, \sigma_2^*, \theta)$

- (P2): $\forall a_1,\ \sigma_2^*(.|a_1) \in \arg\max_{\alpha_2} \sum_{\theta} \mu(\theta|a_1) u_2(a_1, \alpha_2, \theta)$

- (B) $\mu(\theta|a_1) = \dfrac{p(\theta)\sigma_1^*(a_1|\theta)}{\sum_{\theta' \in \Theta} p(\theta')\sigma_1^*(a_1|\theta')}, \quad \text{if } \sum_{\theta' \in \Theta} p(\theta')\sigma_1^*(a_1|\theta') > 0$

  $\mu(.|a1)$, is any prob. distr. on Θ, if $\sum_{\theta' \in \Theta} p(\theta')\sigma_1^*(a_1|\theta') = 0$

Backward induction solution?

- Backward induction was used in games with perfect information
- PBE: strategies are optimal given the beliefs and the beliefs are obtained from equilibrium strategies and observed actions
  - Circularity → PBE cannot be determined by backward induction

An example of a signaling game

- **Two period reputation game**
  - 2 firms on the market

  Period 1: both firms of the market; only firm 1 action a1
  - Actions for firm 1: prey or accommodate
    - If prey: firm 2 gets P2<0
    - If accommodate: firm 2 gets D2>0
  - Type of firm 1: "sane" or "crazy"
    - "crazy" – always prey
    - "sane"
      - If accommodates: payoff for 1: D1>0
      - If preys: payoff for 1: P1< D1 → prefers to accommodate
      - If 2 exits→ M1>D1 (monopoly)

Two period reputation game: cont.

- Period 2 - player 2 selects action a2: stay or exit
  - If exits, it gets a 0 payoff, and player 1 gets M1> D1
- Assumptions:
  - Player 1 knows his type
  - Player 2 believes that player 1 is sane with probability p
  - δ = discount factor between the two periods
- Building reputation for the sane player
  - Player 1 may try to convince player 2 that he is crazy, to get M1>D1 in the second period of the game.

Taxonomy of PBE

- **Separating equilibrium:** the two types of player 1, choose two different actions in period 1
  - Firm 2 has complete information for the second period

$$\mu(\theta = \text{sane} \mid a_1 = \text{accomodate}) = 1$$

$$\mu(\theta = \text{crazy} \mid a_1 = \text{prey}) = 1$$

- **Pooling equilibrium:** the two types of player 1, choose the same action in period 1

$$\mu(\theta = \text{sane} \mid a_1 = \text{prey}) = p$$

- **Hybrid (semi-separating equilibria):** the sane type may randomize between preying and accommodating

$$\mu(\theta = \text{sane} \mid a_1 = \text{prey}) \in (0, p)$$

$$\mu(\theta = \text{sane} \mid a_1 = \text{accommodate}) = 1$$

What type of equilibrium? Existence.

- Separating eq. existence: sufficient and necessary condition

$$D_1(1+\delta) \geq P_1 + \delta M_1 \Leftrightarrow \delta(M_1 - D_1) \leq (D_1 - P_1)$$

- Pooling eq. → to enforce exit for player 2. Condition:

$$pD_2 + (1-p)P_2 \leq 0$$

- If the above two conditions do not hold → hybrid PBE

Note: uniqueness of the eq. in this case, is due to the fact that the “crazy” type is assumed to always prey.

Multi-stage games with observed actions and incomplete information

- Each player i has type θ_i, and types are independent

$$p(\theta) = \prod_{i=1}^{I} p_i(\theta_i)$$

- At each period t (t=0,1,2,… T), players choose their actions simultaneously, and the actions are revealed at the end of the period
- Players' action set at a date t is type independent
- Behavior strategy: $\sigma_i(a_i \mid h^t, \theta)$
- Payoffs $u_i(h^{T+1}, \theta)$

- Subgame perfection → BNE not only for the whole game, but also for the "continuation game" starting at period t after all possible histories h^t
  - Continuation games → proper subgames?
    - No. They do not stem from a singleton information set

Continuation games → true games

- Need to specify the players' beliefs at the start of each continuation game.
- **Definition:** A perfect Bayesian equilibrium is a (σ,μ) that satisfies (P) and (B(i) – B(iv)).

B(i) Posterior beliefs are independent, and all types of player i have the same belief.

- For all θ, t and h^t:

$$\mu_i(\theta_{-i} \mid \theta_i, h^t) = \prod_{j \neq i} \mu_i(\theta_j \mid h^t)$$

- even unexpected events will not change the independence assumption for the type of the opponents