Game theory can be defined as the study of mathematical models of conflict and cooperation between intelligent rational decision-makers. Game theory provides general mathematical techniques for analyzing situations in which two or more individuals make decisions that will influence one another’s welfare.

Roger B. Myerson, 1991

Game theory is a mathematical method for analyzing strategic interaction.

Nobel Prize Citation, 1994
Incomplete Information Games

- Players observe one another actions at the end of each period
- Players do not know the others type
- Start: not a well defined sub-game
- Beliefs must be specified
- Continuation strategies are Nash Equilibrium?

Example: Spence’s Job Market
  - Worker-Leader, knows the productivity
  - Follower-Firms, looks only at the education level not productivity
  - Wage is based on education (not productivity)
Dynamic Games of Incomplete Information

• Recall: static games of incomplete information
  – The game/payoffs depend on the type of players. A player knows its own type but it does not know the types of the other players.
  – Transform a game of incomplete information--game of imperfect information
    • Assign probabilities for the types of the players
    • Perceived as a move by nature
    • Represents the players’ *apriori belief on the types of other players*
• What changes for the dynamic game?
  – Players have the chance of updating their beliefs based on the observed actions of the other players
Incomplete Information: Imperfect Information

- Players actions can convey information to other players
- Including player’s own past actions
Sub-game Perfection for Dynamic Games of Incomplete Information?

- The concept of sub-game perfection: harder to apply for games of incomplete information
  - Start of a period does not form a well-defined sub-game
  - Formally: the only proper sub-game of a game of incomplete information is the whole game--any Nash equilibrium is Sub-game perfect

Example of a game of imperfect information

What are the Nash eq. for this game?
Sub-game Perfection

- Player 1 (L, M, R)
- L: ends the game
- Player plays M or R: player 2 plays A or B
- Player 2 does not know if player 1 chooses M or R
- 2 Pure strategy Nash Equilibrium: (L, A) and (R, B) \{ sub-game perfect equilibrium \}
Perfect Bayesian Equilibrium for Multi-stage Games

- **The Basic Signaling Game (2 players)**
  - Player 1: leader (sender)
  - Player 2: follower (receiver)
  - Player 1 has private info. about its type $\theta \in \Theta$, action $a_1 \in A_1$
  - Player 2: its type is common knowledge, action $a_2 \in A_2$
  - Space of mixed actions: $A_1$ and $A_2$, with elements $a_1$ and $a_2$
  - Utility of player i: $u_i(a_1, a_2, \theta)$
  - Player 2 – prior belief about player’s 1 type: $p \rightarrow$ common knowledge
  - Strategy for player 1: probability distribution $\sigma_1(\cdot | \theta)$ over actions $a_1 \in A_1$ for each type $\theta$
  - Strategy for player 2: probability distribution $\sigma_2(\cdot | \theta)$ over actions $a_2 \in A_2$ for each $a_1 \in A_1$
Payoffs Calculation

• Payoff for player 1, given its type $\theta$:

$$u_1(\sigma_1, \sigma_2, \theta) = \sum_{a_1} \sum_{a_2} \sigma_1(a_1 | \theta) \sigma_2(a_2 | a_1) u_1(a_1, a_2, \theta)$$

• Player 2’s (ex ante – beforehand payoff) to strategy $\sigma_2(. | a_1)$, when player 1 plays $\sigma_1(. | \theta)$:

$$\sum_{\theta} p(\theta) \left[ \sum_{a_1} \sum_{a_2} \sigma_1(a_1 | \theta) \sigma_2(a_2 | a_1) u_2(a_1, a_2, \theta) \right]$$

• Should we make the decision based on the above computed payoff?
Posterior beliefs

• Player 2, observes the action of player 2 $\rightarrow$ must update its belief on $\theta$, and its choice of action $\rightarrow$ posterior distribution over $\Theta$: $\mu(. | a_1)$

• How to compute $\mu(. | a_1)$?
  – Player 1 actions may depend on its type
  – Let $\sigma_1^*(. | \theta)$ to be player 1 strategy
  – Know $p(.), \sigma_1^*(. | \theta)$ and observe $a_1$: use Bayes rule

• Extension of subgame-perfect eq. $\rightarrow$ perfect Bayesian eq.
  – Player 2 max. its payoff conditional on $a_1$:

$$\sum_{\theta} \mu(\theta | a_1) u_2(a_1, \sigma_2(. | a_1), \theta) = \sum_{\theta} \sum_{a_2} \mu(\theta | a_1) \sigma_2(a_2 | a_1) u_2(a_1, a_2, \theta)$$
Perfect Bayesian Equilibrium (PBE)

- **Definition:** A perfect Bayesian eq. of a signaling game is a strategy profile \( \sigma^* \) and posterior beliefs \( \mu(.|a_1) \), s.t.:

- \((P1)\): \( \forall \theta, \sigma^*_1(.|\theta) \in \arg \max_{\alpha_1} u_1(\alpha_1, \sigma^*_2, \theta) \)

- \((P2)\): \( \forall a_1, \sigma^*_2(.|a_1) \in \arg \max_{\alpha_2} \sum_{\theta} \mu(\theta|a_1)u_2(a_1, \alpha_2, \theta) \)

- \((B)\)

\[
\mu(\theta|a_1) = \frac{p(\theta)\sigma^*_1(a_1|\theta)}{\sum_{\theta' \in \Theta} p(\theta')\sigma^*_1(a_1|\theta')}, \quad \text{if } \sum_{\theta' \in \Theta} p(\theta')\sigma^*_1(a_1|\theta') > 0
\]

\( \mu(.|a1) \), is any prob. distr. on \( \Theta \), if \( \sum_{\theta' \in \Theta} p(\theta')\sigma^*_1(a_1|\theta') = 0 \)