Games of Incomplete Information

• Recall:
  – Games of perfect information – sequential games
  – Games of imperfect information – simultaneous move games
  – Games of incomplete information?
    • Some players do not know the payoffs of the others
Bayesian game characterization

- Type of the players \( \rightarrow \) contains any initial private information that a player might have
- Knowledge about the types is characterized by a pdf
  \[
p(\theta_1, \theta_2, ..., \theta_I)
  \]
  \[
p(\theta_{-i} | \theta_i) = \text{player's 1 probability about its opponents types, given its own type}
  \]
  \[
p(\theta_i) > 0, \quad \forall \theta_i \in \Theta_i
  \]
- Given the pure strategy space \( S_i \), the payoff function of each player \( i \), will depend on players’ types:
  \[
u(s_1, ..., s_I, \theta_1, ..., \theta_I)
  \]
Bayesian Equilibrium

- **Definition:** A Bayesian equilibrium in a game of incomplete information with a finite number of types $\theta_i$ for each player $i$, prior distribution $p$, and pure strategy spaces $S_i$ is a Nash equilibrium of the “expanded game”, in which each player’s $i$ space of pure strategies is the set $S_i^{\theta_i}$ of maps from $\theta_i$ to $S_i$.

Define a strategy profile: $s(\cdot)$ and $s'_i \in S_i^{\theta_i}$

The profile $s(\cdot)$ is a pure strategy Bayesian equilibrium if, for each player $i$

\[
s_i(\cdot) \in \arg \max \sum_{s'_i(\cdot) \in S_i} \sum_{\theta_i, \theta_{-i}} p(\theta_i, \theta_{-i}) u_i(s'_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i, \theta_{-i})
\]

\[
s_i(\cdot) \in \arg \max \sum_{s'_i(\cdot) \in S_i} \sum_{\theta_{-i}} p(\theta_{-i} | \theta_i) u_i(s'_i, s_{-i}(\theta_{-i}), \theta_i, \theta_{-i})
\]
Example: Cournot competition

- Firms select the quantities of production $s_i = q_i$
- Payoffs defined as
  \[ u_i = q_i(\theta_i - q_i - q_j) \]
- Common knowledge: firm 1 has type $\theta_1 = 1$
- Firm 2 – private information about $\theta_2$
- Firm one beliefs:
  \[ \theta_2 = \frac{3}{4}, \quad p = \frac{1}{2} \]
  \[ \theta_2 = \frac{5}{4}, \quad p = \frac{1}{2} \]
- Belief of Firm 1 is common knowledge
- Firms choose their outputs simultaneously
Cournot competition: equilibrium

- $q_1 =$ firm ones output
- For firm 2: for $\theta_2 = 3/4$, $q_2^L$
  
  $\theta_2 = 5/4$, $q_2^H$
  
  $\frac{\partial u_2}{\partial q_2} = 0 \Rightarrow q_2(\theta_2) = \frac{\theta_2 - q_1}{2}$

\[
\begin{align*}
\frac{\partial u_1}{\partial q_1} = 0 & \Rightarrow \frac{1}{2} q_1 (1 - q_1 - q_2^H) + \frac{1}{2} q_1 (1 - q_1 - q_2^L) = 0 \Rightarrow q_1 = \frac{2 - q_2^H - q_2^L}{4}
\end{align*}
\]

Unique Bayesian equilibrium: $(q_1 = 1/3; \quad q_2^L = 11/24; \quad q_2^H = 5/24)$
Example

- An industry with 2 firms: incumbent (player 1) and potential entrant (player 2)
  - Player 1: Build new plant?
  - Player 2: Enter?

<table>
<thead>
<tr>
<th>Build</th>
<th>Enter</th>
<th>Don’t</th>
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<tbody>
<tr>
<td></td>
<td>0, -1</td>
<td>2, 0</td>
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Building cost HIGH

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Building cost LOW

Player 1 knows its cost for building (HIGH or LOW)
Player 2 does not know
→ assign a probability p1 for HIGH
Equivalent game

- Introduce prior move by nature: determines player 1 “type”
- Player’s 2 incomplete information $\rightarrow$ imperfect information
Equilibrium solution

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Building cost HIGH

If cost is HIGH: Don’t build $\rightarrow$ dominant strategy for player 1

Building cost LOW

If cost is LOW; $\rightarrow$ dominant strategy for player 1: build

How about player 2?
- strategy for 2:

$$-1 \times (1 - p_1) + 1 \times p_1 > 0 \Rightarrow p_1 > 1/2$$
## Equilibrium solution

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<td>2,0</td>
<td>Build</td>
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### Building cost HIGH

- If cost is HIGH: Don’t build → dominant strategy for player 1

### Building cost LOW

- If cost is LOW; no dominant strategy
  - $y = \text{probability that player 2 enters}$
  - $x = \text{probability that player 1 builds (given the type of player 1)}$

### Analysis

- Building better than not building: $1.5y + 3.5(1-y) > 2y + 3(1-y) \Rightarrow y < \frac{1}{2}$
- Enter better than not enter: 
  - $(-1)x(1-p_1) + 1[1-x(1-p_1)] > 0$
  - $\Rightarrow x < 1/[2(1-p_1)]$