Game Theory

Department of Electronics

EL-766

Spring 2011

Lecture 5

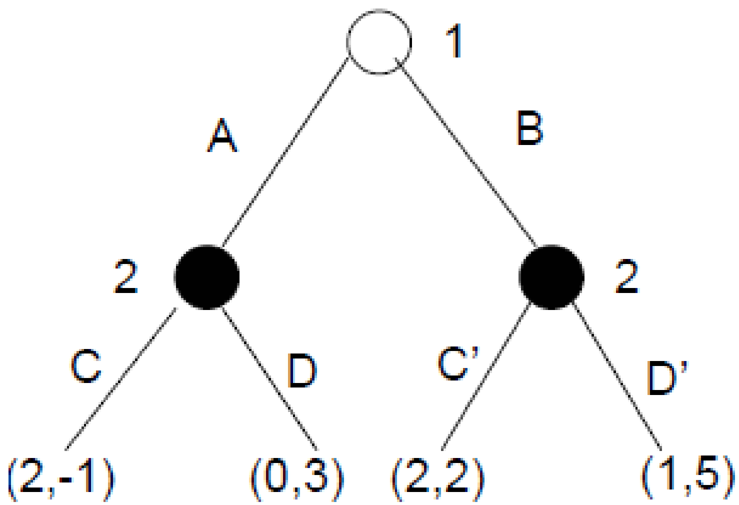
Mixed strategies

- Behavior strategies
- A behavior strategy specifies a probability distribution over actions at each h_i, and the probability distributions at different information sets are independent.

- Nash eq. in behavior strategies = profile such that no player can increase its expected payoff by using a different behavior strategy.
- Mixed strategies and behavior strategies are equivalent for games of perfect recall
  - No player ever forgets any information he knew
  - Library analogy for mixed and behavior strategies (ex. In book)

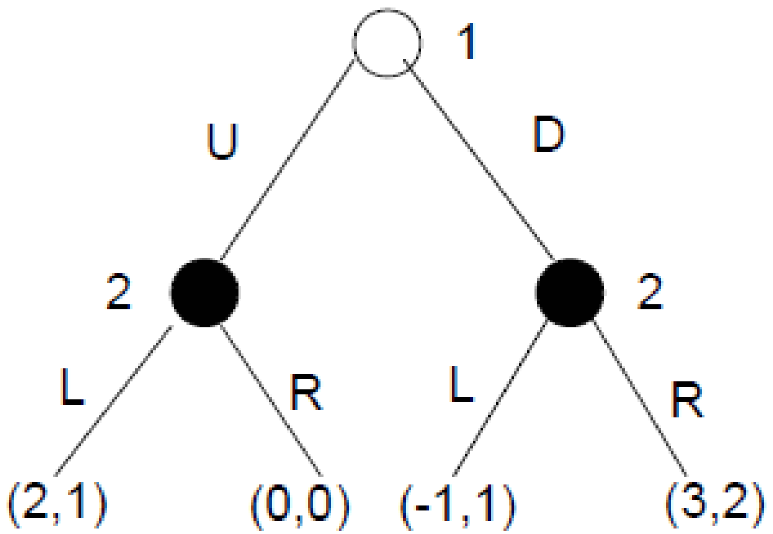
Equivalence with games in strategic form

- Basic idea: represent every strategy and their corresponding payoff:
- Consider an example game
  - Player 1: {A,B}
  - Player 2: {(C,C'),(C,D'),(D,C'),(D,D')}



	C,C'	C,D'	D,C'	D,D'
A	2,-1	2,-1	0,3	0,3
B	2,2	1,5	2,2	1,5

Another example



Find strategic form equivalence
Find Nash equilibria

Nash equilibria

- If the extensive form is finite → the corresponding strategic form is finite → Nash theorem guarantees the existence of a mixed-strategy equilibrium
- Iterative strict dominance notion extends for extensive form games as well
  - Weaker notion: a player cannot strictly prefer one action over another at an information set that is not reached, given its opponent's play

- Theorem (Zermelo, Kuhn): **A finite game of perfect information has a pure strategy Nash equilibrium.**

  - Proof based on many player generalization of backward induction in dynamic programming
  - Idea: the game finite → has a set of penultimate nodes → players moving at these nodes chooses strateg that leads to terminal node with max. payoff. Players that have successors the penultimate nodes, choose actions that max. their payoffs, given the choice of the penultimate nodes. Etc… Roll back to the three → resulting strategy is a Nash eq.

Nash equilibria comments

- Previous theorem may not hold if the hypothesis are weakened
  - Infinite games
    - Node with an infinite number of successors: continuum of actions
    - Path with an infinite number of nodes: multi-stage games with an infinite number of stages
  - Not perfect information (i.e., some of the information sets are not singletons)

Backward induction and subgame perfection

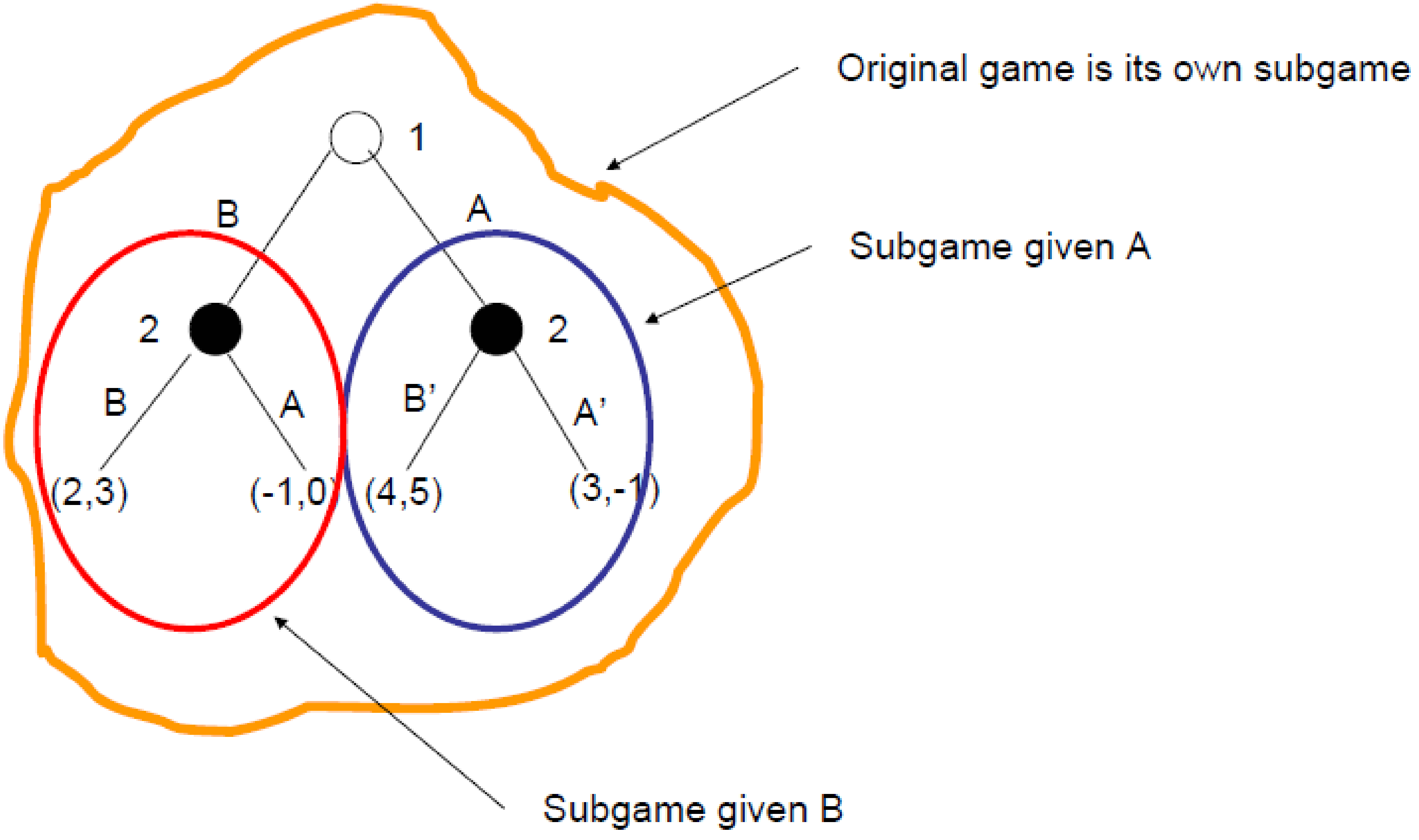
- Find Nash eq. by backward induction:
  - Reason backwards on what each rational player would play
  - Assumption: Starting at any decision point in the game, a player's strategy (from that point on) is a best response to the strategies of other players → Sequential Rationality
  - Subgame perfect Nash equilibrium → key concept for the backward induction technique

Subgame

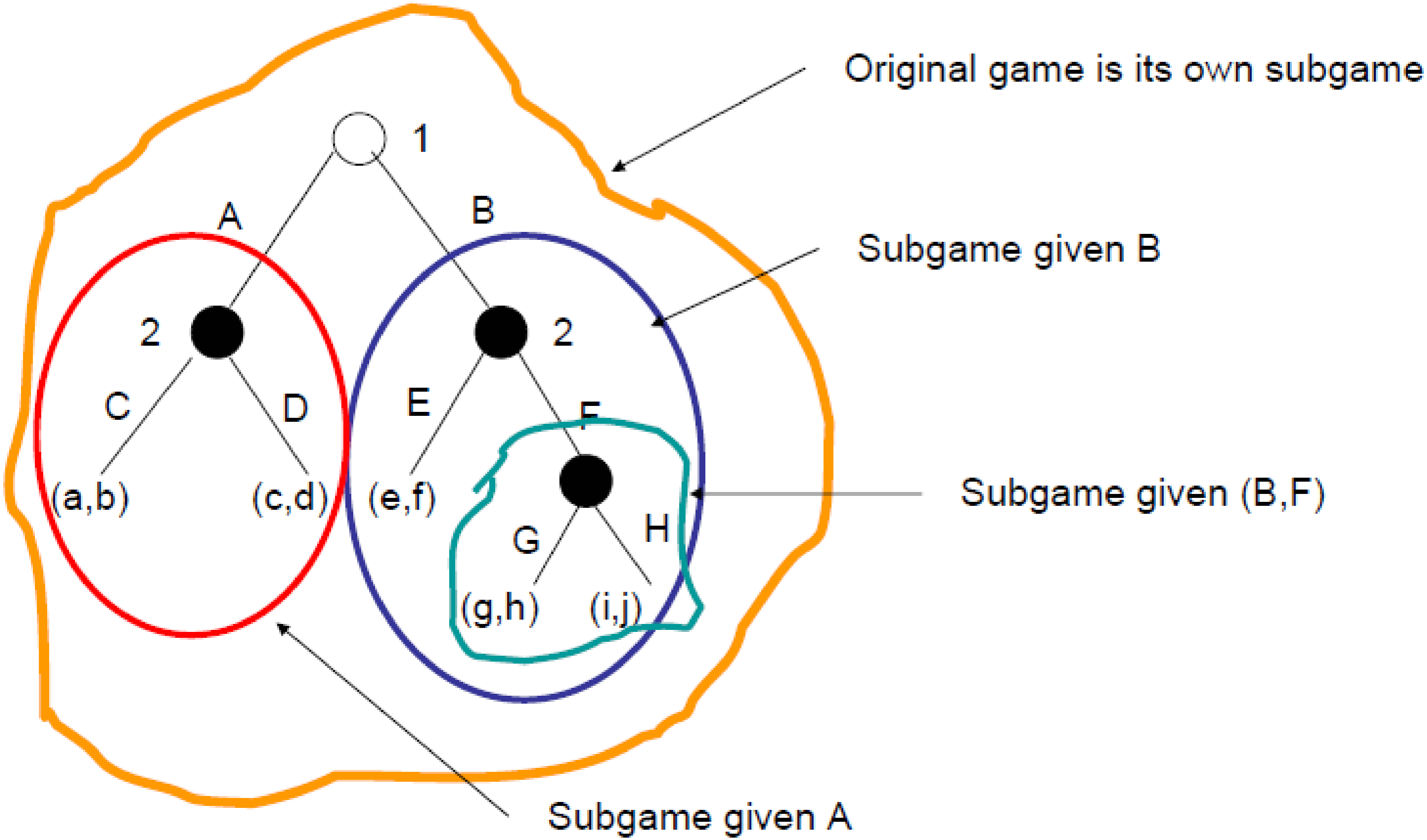
- Definition: A proper subgame G of an extensive form game T, consists of a single node and all of its successors in T, with the property that if $x' \in G$, and $x'' \in h(x')$, then $x'' \in G$. The information sets and payoffs are inherited from the original game.

Simplified: A subgame is determined by assuming that h has already happened, and selecting the game from that point onward.

Subgame example



Another subgame example



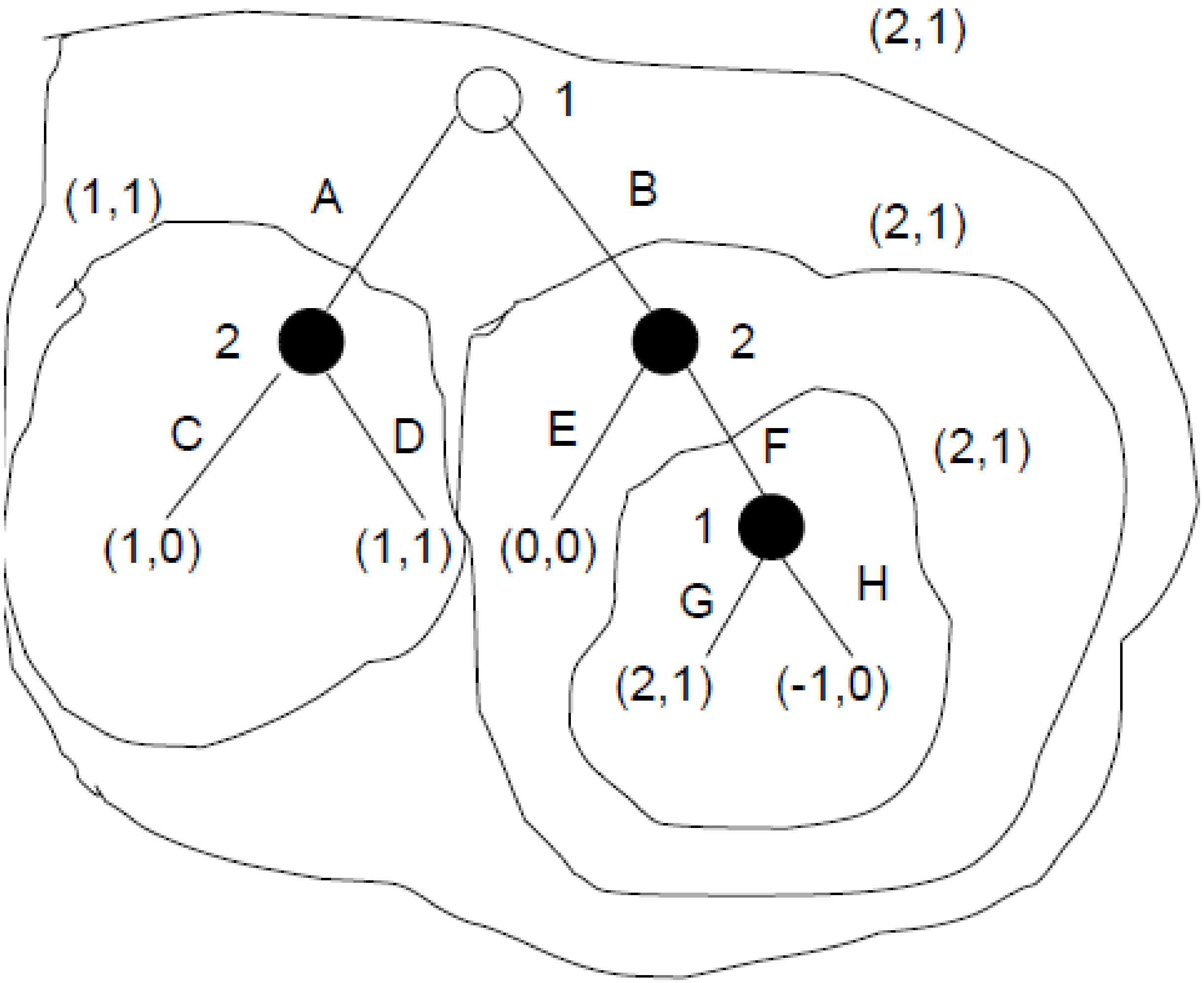
Subgame perfect equilibrium

- **Definition:** A behavior-strategy profile σ of an extensive-form game is a subgame-perfect equilibrium, if the restriction of σ to G is a Nash equilibrium of G for every proper subgame G.

Backward induction algorithm

- Identify all terminal subgames
- Determine the Nash eq. for these subgames
- Modify the original game tree by replacing the terminal subgames with the Nash equilibrium payoffs
- Repeat until the three is reduced to one stage game, and then determine the Nash equilibrium.

Example backward induction algorithm



{B,F,G} = Nash equilibrium

Payoff is (2,1)

Subgame perfection and one stage deviation principle

- **Finite horizon games:** In a finite multistage game with observed actions, a strategy profile s is subgame perfect, iff no player can gain by deviating from s in a single stage and then conforming to s thereafter.
- **Infinite horizon games:** In an infinite multistage game with observed actions that is continuous at infinity (events in distant future are relatively unimportant), profile s is subgame perfect, iff there is no player i and strategy $s^\wedge_I$ that agrees with si except at a single t and h^t, such that $s^\wedge_I$ is a better response to s_{-I} , than si, conditioned on history h^t being reached.

Repeated games

- Introduces new equilibria: players may condition their actions on the way their opponents play in previous periods.
- Example: prisoner's dilemma
- Payoffs depend only on current actions ($g_i(a^t)$ shown in matrix)
- Players discount future payoffs with a common discount factor δ
- Questions: how the eq. payoffs vary with the horizon T, and the discount factor ?

	C	D
C	1,1	-1,2
D	2,-1	0,0

Repeated games-cont.

- Cummulative payoff
- Normalized – to make it comparable for different time horizons T
- The utility of a sequence

$$\left\{a^0, a^1, \ldots, a^T\right\}$$

$$\frac{1-\delta}{1-\delta^{T+1}} \sum_{t-0}^{T} \delta^t g_i\left(a^t\right) \longleftarrow \text{Average discounted payoff}$$

Prisoners' dilemma example

- Game played only once: equilibrium - both defect
- Game repeated a finite # of times
  - Both defect remains a sub-game perfect equilibrium
  - If horizon is infinite, and $\delta > ½$ → new subgame perfect eq.:
    - "Cooperate in the first period and continue to cooperate as long as no player has ever defected. If any player has ever defected, then defect for the rest of the game".
  - Two classes of subgames: A – no player has defected
  - B – defect i has occurred
- If player i conforms to A for all stages of the game : payoff is 1
- If deviates at time t: its normalized payoff is

$$(1-\delta)(1+\delta+\ldots+\delta^{t-1}+2\delta^{t}+0+0+\ldots)=1-\delta^{t}(2\delta-1)$$

New equilibrium for Prisoners' dilemma

- In every subgame, no player can gain by deviating once from the specified strategy and then conforming → one stage deviation principle holds → the strategies form a subgame perfect eq.
- Depending on the size of the discount factor, there can exist many other perfect equilibria
- Next time: we will discuss folk theorem
  - Repeated play with patient players not only makes cooperation possible (more efficient payoffs) but it leads to a large set of other equilibrium outcomes.