Dynamic games of complete information

• Extensive form games
  – In the examples studied so far, players choose their actions simultaneously
  – Can the strategic form game model the situation in which the order in which players move influences the outcome of the game?
  – Recall strategic (normal) form game characterized by three elements:
    • The set of players: \{1, 2, ..., I\} (finite set)
    • The pure strategy space for each player i: \(S_i\)
    • Payoff (utility functions) for each profile of strategies: \(s = (s_1, ..., s_I)\)
Example: Cournot vs. Stackelberg equilibrium

- Actions: choices of output levels: \( q_1 \) and \( q_2 \)
- Cournot: both players choose their actions simultaneously, i.e., try to simultaneously maximize their utility functions
  - Example:
    
    \[
    u_i(q_1, q_2) = [12 - (q_1 + q_2)]q_i \Rightarrow \begin{cases} 
    \frac{\partial u_1(q_1, q_2)}{\partial q_2} = 0 \Rightarrow r_1(q_2) = 6 - \frac{q_2}{2} \\
    \frac{\partial u_2(q_1, q_2)}{\partial q_1} = 0 \Rightarrow r_2(q_1) = 6 - \frac{q_1}{2}
    \end{cases}
    \]

- Stackelberg: player 1 chooses first, then player 2 observes the output \( q_1 \), and consequently chooses \( q_2 \)
  - Is it the same equilibrium?
  - For which player this game is more advantageous?
Stackelberg equilibrium: cont.

- Player 2 sees $q_1$, computes $r_2(q_1)$ in the same fashion as before.
- Player 1: knows that player 2 will maximize its utility based on $q_1$, can compute $r_2(q_1)$, and then maximize its utility by appropriately selecting $q_1$.

$$u_i(q_1, q_2) = [12 - (q_1 + q_2)]q_i \Rightarrow \begin{cases} \frac{\partial u_2(q_1, q_2)}{\partial q_2} = 0 \Rightarrow r_2(q_1) = 6 - \frac{q_1}{2} \\ u_1(q_1, q_2) = q_1 \left(6 - \frac{q_1}{2}\right) \\ \frac{\partial u_1(q_1)}{\partial q_1} = 0 \Rightarrow q_1^* = 6 \end{cases}$$
Stackelberg equilibrium: cont

- The resulting equilibrium point: $q^*_{1}=6$, $q^*_{2}=3$, with payoffs $(18,9)$
- Obtained by backward induction
- Cournot equilibrium: $q^C_{1}=4$, $q^C_{2}=4$, with payoffs $(16,16)$

- Leader has the advantage
- Other possible equilibria?
- Maybe, but not credible: would rely on empty threats from player 2, to maintain a different level $q_2$
Extensive form games

- Stackelberg game (leader-follower) – example of a game in which players move sequentially and the order of the players’ moves matters
- Multi-stage game
- Game of perfect information: exactly one player moves at a given stage, all the others have the one element choice: “do nothing”
- What is an extensive form game?
Extensive form games

The extensive form of a game contains the following info:

- The set of players $i \in I$
- The order of moves: Game tree
- The players’ payoffs as a function of the previous moves
- What are the players choices when they move
- What each player knows when he makes its choice
- The probability distribution over any exogenous events

Exogenous events: moves by nature
Characterizing previous moves: definitions and notations

- Multi-stage game:
  - Players move simultaneously at stage $k$ (do not know the actions of their opponents for stage $k$)
  - Know all the actions chosen at previous stages: $0, 1, 2, \ldots, k-1$.
  - Particular case: Stackelberg example (2 stage game) – at one stage just one player moves, the other one has action “do nothing”.

- At stage $k$, $i$-th player chooses an action from the choice set $A_i(h^k)$

$$h^k = (a_0, a_1, \ldots, a^k)$$

= the history at the end of stage $k$

$$a^k = (a_1^k, a_2^k, \ldots, a_I^k)$$

= stage $k$ strategy profile

Game begins at stage 0, with $h^0 = \emptyset$
More notations

- $H^k$ = the set of all stage $k$ histories
- $Z$ = the set of terminal histories

- Player’s i payoff represented as
  \[ u_i : H^K \rightarrow R \]

- For many applications: the payoff is additive over the game stages (weighted sum)

- Single stage payoffs for user i at stage k: \[ g_i(a^k) \]
Game tree representation

- Components of a game tree
  - Vertices (nodes) – represent a particular history
    - Usually represented as ●
    - Exception: node with no history (first stage) ○
    - Labeled with the user id
  - Edges: correspond to the actions taken
    - Labeled with the action element
  - Simultaneous moves can also be modeled
  - When a node uncertain of past history (particular case is simultaneous moves, a dashed line unites its vertices for that stage. Sometimes this is represented also by encircling the vertices with an ellipse
Game tree representation – cont.

- Time progresses in one direction: typically from left to right, or top to bottom.
Stackelberg game representation

- Simplifying assumption: each player has only three possible output levels: 3, 4, 6

![Game representation diagram]
Cournnot equivalent

- Is this representation unique?
Strategies and equilibria for extensive form games

- $H_i =$ player $i$ information sets
- $A_i = \bigcup_{h_i \in H_i} A(h_i) =$ set of all actions for player $i$
- A pure strategy is a map
  \[ s_i : H_i \rightarrow A_i, \quad s_i(h_i) \in A(h_i), \quad \forall h_i \in H_i \]
- Player $i$ pure strategy space, is the space of all $s_i$
- The number of player’s $i$ pure strategy:
  \[ \# S_i = \prod_{h_i \in H_i} \# A(h_i) \]
- Path of $s =$ information sets that are reached with positive probability
- Pure strategy Nash equilibrium: a strategy profile $s^*$, such that each player’s $i$ strategy ($s_i^*$) maximizes his expected payoff, given the strategies of his opponents ($s_{-i}^*$)
Stackelberg game example

- How many pure strategies for players?

Payoffs for each strategy profile