Game Theory

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EL-766

Fall 2015

Outline

- Dynamic games of incomplete information
  - Equilibrium refinements
    - Perfect Bayesian equilibrium
    - Sequential equilibrium
- Learning in games
  - Fictitious play

Game theory can be defined as the study of mathematical models of conflict and cooperation between intelligent rational decision-makers. Game theory provides general mathematical techniques for analyzing situations in which two or more individuals make decisions that will influence one another's welfare.

Roger B. Myerson, 1991

Game theory is a mathematical method for analyzing strategic interaction.

Nobel Prize Citation, 1994

Incomplete Information Games

- Players observe one another actions at the end of each period
- Players do not know the others type
- Start: not a well defined sub-game
- Beliefs must be specified
- Continuation strategies are Nash Equilibrium?

Example: Spence's Job Market

- Worker-Leader, knows the productivity
- Follower-Firms, looks only at the education level not productivity
- Wage is based on education (not productivity)

Dynamic Games of Incomplete Information

- Recall: static games of incomplete information
  - The game/payoffs depend on the type of players. A player knows its own type but it does not know the types of the other players.
  - Transform a game of incomplete information--game of imperfect information
    - Assign probabilities for the types of the players
    - Perceived as a move by nature
    - Represents the players' *apriori belief on the types of other* players
- What changes for the dynamic game?
  - Players have the chance of updating their beliefs based on the observed actions of the other players

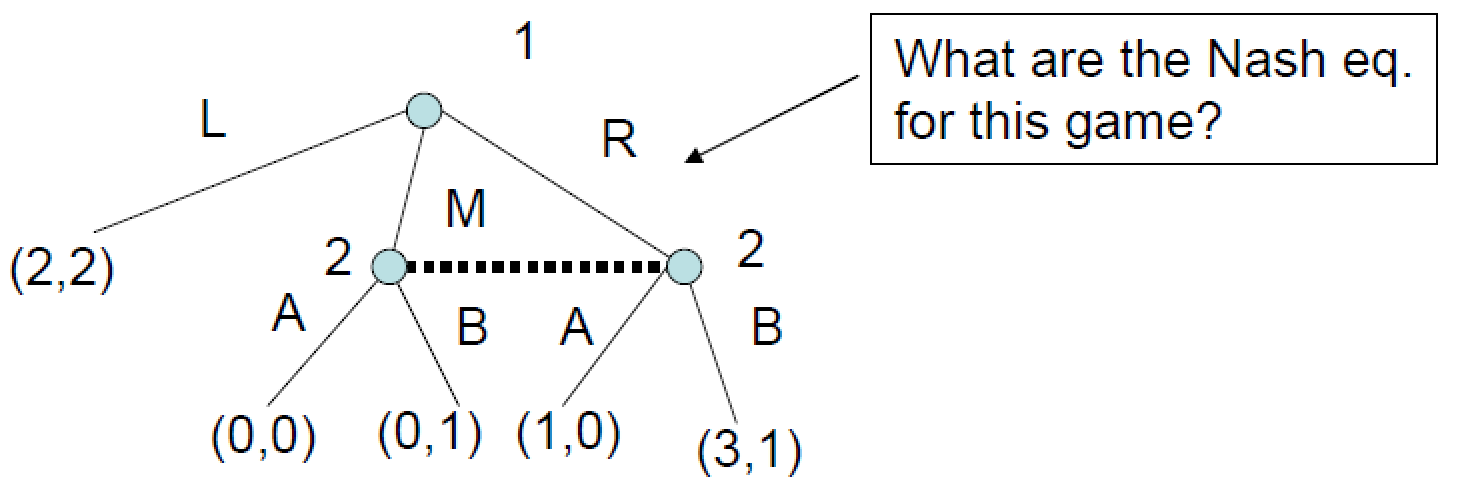
Incomplete Information: Imperfect Information

- Players actions can convey information to other players
- Including player’s own past actions

Sub-game Perfection for Dynamic Games of Incomplete Information?

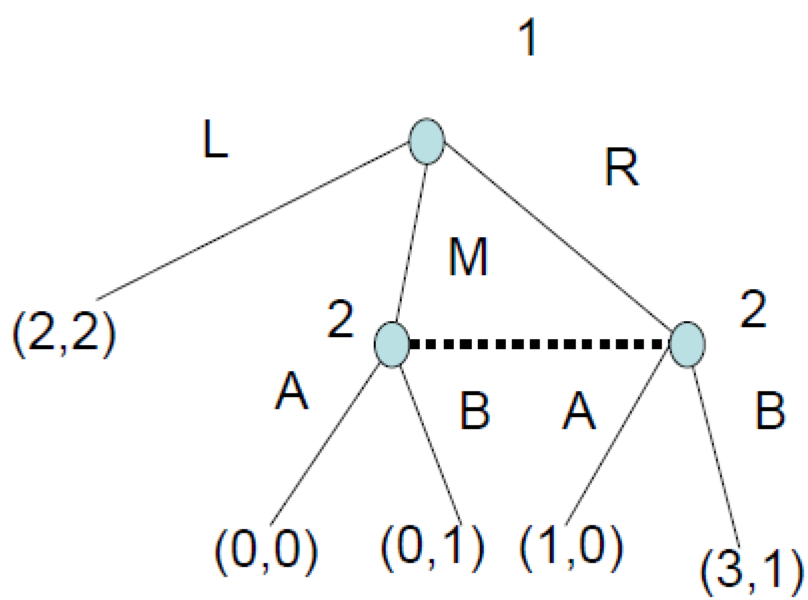
- The concept of sub-game perfection: harder to apply for games of incomplete information
  - Start of a period does not form a well-defined sub-game
  - Formally: the only proper sub-game of a game of incomplete information is the whole game--any Nash equilibrium is Sub-game perfect

Example of a game of imperfect information



Sub-game Perfection

- Player 1 (L,M,R)
- L: ends the game
- Player plays M or R: player 2 plays A or B
- Player 2 does not know if player 1 choose M or R
- 2 Pure strategy Nash Equilibrium: (L, A) and (R, B) { sub-game perfect equilibrium}



Perfect Bayesian Equilibrium for Multi-stage Games

- **The Basic Signaling Game (**2 players)
  - Player 1: leader (sender)
  - Player 2: follower (receiver)
  - Player 1 has private inf. about its type $\theta \in \Theta$, action $a_1 \in A_1$
  - Player 2: its type is common knowledge, action $a_2 \in A_2$
  - Space of mixed actions: $\mathcal{A}_1$ and $\mathcal{A}_2$, with elements α_1 and α_2
  - Utility of player i: $u_i(\alpha_1, \alpha_2, \theta)$
  - Player 2 – prior belief about player's 1 type: p → common knowledge
  - Strategy for player 1: probability distribution $\sigma_1(.| \theta)$ over actions $a_1 \in A_1$ for each type θ
  - Strategy for player 2: probability distribution $\sigma_2(.| \theta)$ over actions $a_2 \in A_2$ for each $a_1 \in A_1$

Payoffs Calculation

- Payoff for player 1, given its type θ:

$$u_1(\sigma_1, \sigma_2, \theta) = \sum_{a_1} \sum_{a_2} \sigma_1(a_1 \mid \theta) \sigma_2(a_2 \mid a_1) u_1(a_1, a_2, \theta)$$

- Player 2's (ex ante – beforehand payoff) to strategy $\sigma_2(.|a_1)$, when player 1 plays $\sigma_1(.|\,\theta)$:

$$\sum_{\theta} p(\theta) \left[\sum_{a_1} \sum_{a_2} \sigma_1(a_1 \mid \theta) \sigma_2(a_2 \mid a_1) u_2(a_1, a_2, \theta) \right]$$

- Should we make the decision based on the above computed payoff?

Posterior beliefs

- Player 2, observes the action of player 2 → must update its belief on θ, and its choice of action → posterior distribution over Θ: $\mu(.|a_1)$
- How to compute $\mu(.|a_1)$?
  - Player 1 actions may depend on its type
  - Let $\sigma_1^*(.|\theta)$ to be player 1 strategy
  - Know p(.), $\sigma_1^*(.|\theta)$ and observe a_1: use Bayes rule
- Extension of subgame-perfect eq. → perfect Bayesian eq.
  - Player 2 max. its payoff conditional on a_1:

$$\sum_{\theta} \mu(\theta \mid a_1) u_2(a_1, \sigma_2(. \mid a_1), \theta) = \sum_{\theta} \sum_{a_2} \mu(\theta \mid a_1) \sigma_2(a_2 \mid a_1) u_2(a_1, a_2, \theta)$$

Perfect Bayesian Equilibrium (PBE)

- **Definition:** A perfect Bayesian eq. of a signaling game is a strategy profile σ^* and posterior beliefs $\mu(.|a_1)$, s.t.:

- (P1): $\forall \theta,\ \sigma_1^*(.|\theta) \in \arg\max_{\alpha_1} u_1(\alpha_1, \sigma_2^*, \theta)$

- (P2): $\forall a_1,\ \sigma_2^*(.|a_1) \in \arg\max_{\alpha_2} \sum_{\theta} \mu(\theta|a_1) u_2(a_1, \alpha_2, \theta)$

- (B) $$\mu(\theta|a_1) = \frac{p(\theta)\sigma_1^*(a_1|\theta)}{\sum_{\theta' \in \Theta} p(\theta')\sigma_1^*(a_1|\theta')}, \quad \text{if} \quad \sum_{\theta' \in \Theta} p(\theta')\sigma_1^*(a_1|\theta') > 0$$

  $\mu(.|a1)$, is any prob. distr. on Θ, if $\sum_{\theta' \in \Theta} p(\theta')\sigma_1^*(a_1|\theta') = 0$