Game Theory

Department of Electronics

EL-766

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Bayesian game characterization

- Type of the players → contains any initial private information that a player might have
- Knowledge about the types is characterized by a pdf

$$p(\theta_1, \theta_2, \ldots, \theta_I)$$

$p(\theta_{-i} \mid \theta_i)$= player's 1 probability about its opponents types, given its own type

$$p(\theta_i) > 0, \quad \forall \theta_i \in \Theta_i$$

- Given the pure strategy space S_i, the payoff function of each player i, will depend on players' types:

$$u(s_1, \ldots, s_I, \theta_1, \ldots, \theta_I)$$

Bayesian Equilibrium

- **Definition:** A Bayesian equilibrium in a game of incomplete information with a finite number of types θ_i for each player i, prior distribution p, and pure strategy spaces Si is a Nash equilibrium of the “expanded game”, in which each player’s i space of pure strategies is the set $S_i^{\theta_i}$ of maps from θ_i to S_i.

Define a strategy profile: s(.) and $s'_i \in S_i^{\theta_i}$
The profile s(.) is a pure strategy Bayesian equilibrium if, for each player i

$$s_i(.) \in \arg\max_{s'_i(.) \in S_i} \sum_{\theta_i} \sum_{\theta_{-i}} p(\theta_i, \theta_{-i}) u_i(s'_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i, \theta_{-i})$$

$$s_i(.) \in \arg\max_{s'_i(.) \in S_i} \sum_{\theta_{-i}} p(\theta_{-i} \mid \theta_i) u_i(s'_i, s_{-i}(\theta_{-i}), \theta_i, \theta_{-i})$$

Example: Cournot competition

- Firms select the quantities of production si = qi
- Payoffs defined as

$$u_i = q_i(\theta_i - q_i - q_j)$$

- Common knowledge: firm 1 has type $\theta_1 = 1$

- Firm 2 – private information about θ_2
- Firm one beliefs:

$$\theta_2 = 3/4, \quad p = 1/2$$
$$\theta_2 = 5/4, \quad p = 1/2$$

- Belief of Firm 1 is common knowledge
- Firms choose their outputs simultaneously

Cournot competition: equilibrium

- q_1 = firm ones output
- For firm 2: for $\theta_2 = 3/4, \quad q_2^L$

  $\theta_2 = 5/4, \quad q_2^H$

$$\frac{\partial u_2}{\partial q_2} = 0 \Rightarrow q_2(\theta_2) = \frac{\theta_2 - q_1}{2}$$

$$\frac{\partial u_1}{\partial q_1} = 0 \Rightarrow \frac{1}{2} q_1(1 - q_1 - q_2^H) + \frac{1}{2} q_1(1 - q_1 - q_2^L) = 0 \Rightarrow q_1 = \frac{2 - q_2^H - q_2^L}{4}$$

Unique Bayesian equilibrium: $(q_1 = 1/3;\ q_2^L = 11/24;\ q_2^H = 5/24)$

Example

- An industry with 2 firms: incumbent (player 1) and potential entrant (player 2)
  - Player 1: Build new plant ?
  - Player 3: Enter?

	Enter	Don't
Build	0,-1	2,0
Don't	2,1	3,0

Building cost HIGH

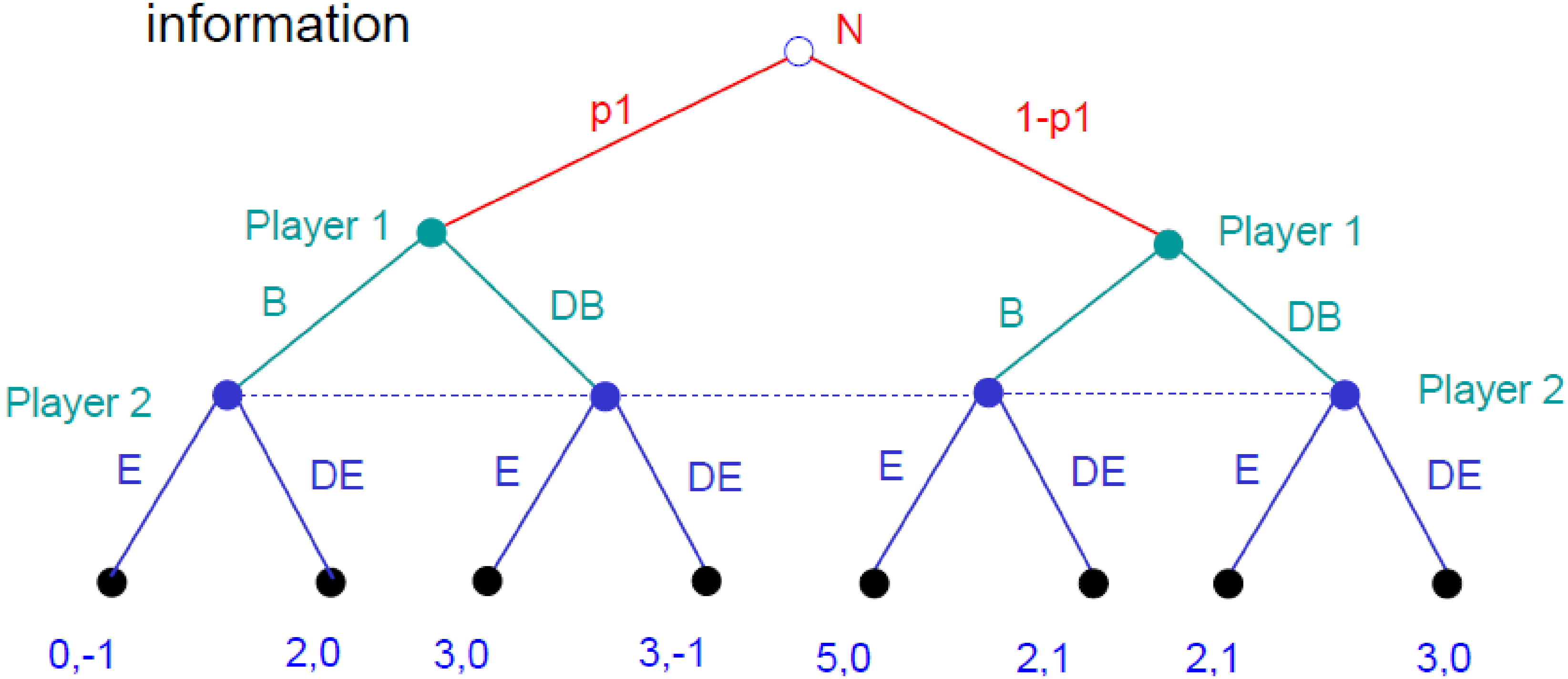
	Enter	Don't
Build	3,-1	5,0
Don't	2,1	3,0

Building cost LOW

Player 1 knows its cost for building (HIGH or LOW)
Player 2 does not know
→ assign a probability p1 for HIGH

Equivalent game

- Introduce prior move by nature: determines player 1 “type”
- Player’s 2 incomplete information → imperfect information



Equilibrium solution

	Enter	Don't
Build	0,-1	2,0
Don't	2,1	3,0

Building cost HIGH

	Enter	Don't
Build	3,-1	5,0
Don't	2,1	3,0

Building cost LOW

If cost is HIGH: Don't build → dominant strategy for player 1

If cost is LOW; → dominant strategy for player 1: build

How about player 2?

- strategy for 2:

$$\text{enter if} : -1 * (1 - p_1) + 1 * p_1 > 0 \Rightarrow p_1 > 1/2$$

Equilibrium solution

	Enter	Don't
Build	0,-1	2,0
Don't	2,1	3,0

Building cost HIGH

	Enter	Don't
Build	1.5,-1	3.5,0
Don't	2,1	3,0

Building cost LOW

If cost is HIGH: Don't build → dominant strategy for player 1

If cost is LOW; no dominant strategy
→ y = probability that player 2 enters
→ x = probability that player 1 builds (given the type of player 1)

Building better than not building: $1.5y+3.5(1-y) > 2y+3(1-y) \rightarrow y< ½$
Enter better than not enter: $(-1)\ x(1-p_1) + 1[1-x(1-p_1)]>0$
$\rightarrow x<1/[2(1-p_1)]$

Another Example: Ballet or Football

Game statement for the imperfect information case:

- Alice and Bob must individually choose to attend either ballet or a football event in the afternoon.
- Common knowledge:
  - ❖ Both would like to spend the afternoon together.
  - ❖ Alice prefers the Ballet.
  - Bob prefers Football.

		Bob: B	Bob: F
Alice	B	2,1	0,0
	F	0,0	1,2

With Incomplete Information

- Bob's preference depends on whether he is happy or not.
- If he is happy → prefers to attend the same event as Alice
- If he is unhappy → prefers to spend the evening alone
- Alice doesn't know if Bob is happy or not. Believes that Bob is happy with probability 0.5 and unhappy with probability 0.5

Bob unhappy

		Bob: B	Bob: F
Alice	B	2,0	0,2
	F	0,1	1,0

Incomplete Information

- What is the Bayesian Nash equilibrium?

Bob happy

		Bob: B	Bob: F
Alice	B	2,1	0,0
	F	0,0	1,2

Bob unhappy

		Bob: B	Bob: F
Alice	B	2,0	0,2
	F	0,1	1,0

Incomplete Information: Continued

- Best response
  - If Alice chooses B → Bob's best response: B if he is happy, and F if he is unhappy
  - After Bob's choice, what is the best response for Alice?
    - Expected payoff for Alice if her choice is B: $2 \times 0.5 + 0 \times 0.5 = 1$
    - Expected payoff for Alice if her choice is F: $0 \times 0.5 + 1 \times 0.5 = 0.5$
    - Best response for Alice is B
  - (B, (B if happy and F if unhappy)) is a Bayesian Nash equilibrium

Incomplete Information: Continued

- Best response
  - If Alice chooses F then Bob's best response: F if he is happy, and B if he is unhappy
  - After Bob's choice, what is Alice's best response?
    - Alice expected payoff if she chooses B:
      $0 \times 0.5 + 2 \times 0.5 = 1$
    - Alice expected payoff if she chooses F
      $1 \times 0.5 + 0 \times 0.5 = 0.5$
    - Since 1>0.5, Alice's best response is B
  - (F, (F if happy and B if unhappy)) is not a Bayesian Nash equilibrium.

Wireless Communications

- “Wireless Communications: Principles and Practice”, T.S. Rappaport, December 2001, Prentice Hall