Game Theory

Department of Electronics

EL-766

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- Repeated Games
- Infinitely Repeated Games
- Folk Theorem
- Strategies for Repeated Games
- Incomplete Information Games
- Bayesian Games

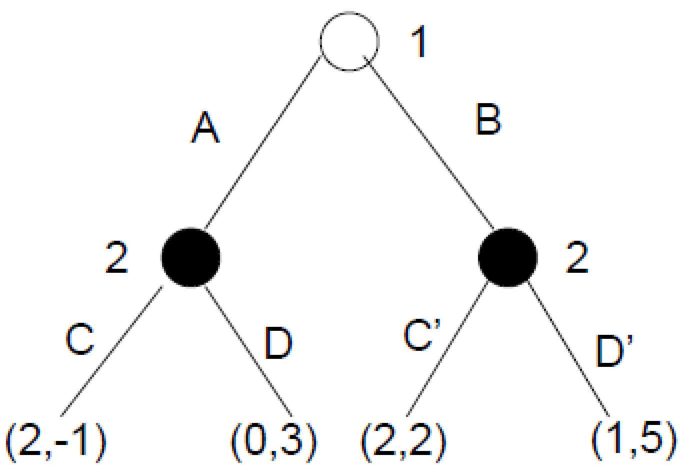
Repeated Games

- Review: repeated games may introduce new equilibrium points and may motivate players for cooperation

- Finitely Repeated Games
  - Backward Induction
- Finite Horizon Games
- Infinite Horizon Games

Alternate representation
Extensive Form of Games: Review

- Equivalence between extensive form game and normal form games



contingent strategies for player 2

	C,C'	C,D'	D,C'	D,D'
A	2,-1	2,-1	0,3	0,3
B	2,2	1,5	2,2	1,5

Folk Theorem for Infinitely Repeated Games

- If players are sufficiently patient, then any feasible, individually rational payoffs can be enforced by an equilibrium.
- In the limit of extreme patience, repeated play allows virtually any payoff to be an equilibrium outcome.

Theorem: For every feasible payoff vector v with $v_i > \underline{v}_i$ for all players i, there exist $\underline{\delta} < 1$, such that for all $\delta \in (\underline{\delta}, 1)$, there is a Nash equilibrium of $G(\delta)$ with payoffs v.

$$\underline{v}_i = \min_{\alpha_{-i}} \left[\max_{\alpha_i} g_i(\alpha_i, \alpha_{-i}) \right]$$

Folk Theorem II

- v_i-player i's reservation utility of minmax value.
- This is the lowest payoff player i's opponents can hold him to by any choice of alpha

Some definitions: $\underline{v}_i$ = the lowest payoff, players' i opponents can hold him to

$$m_{-i}^{i} = \text{minimax profile against player i}$$

$$V = \text{convex hull}\{v \mid \exists\, a \in A, \text{with } g(a) = v\}$$

Proof idea: based on "unrelenting strategy", a player who deviates will be minmaxed in every subsequent period.

- Assume there exists a pure action profile a, s.t. g(a)=v
- Strategy of players: play a if action a was played in the previous period, or the action played differed in two or more components; if in the previous period, player one was the only one to deviate, then play m_j^i for the rest of the game
- Question: can player i gain by deviating from this strategy profile?

Folk Theorem III

If player 1 deviates at period t:

$$(1-\delta^t)v_i + \delta^t(1-\delta)\max_a g_i(a) + \delta^{t+1}\underline{v}_i$$

This payoff is less than v_i, if $\delta > \underline{\delta}_i$

$$(1-\underline{\delta}_i)\max_a g_i(a) + \underline{\delta}_i \underline{v}_i = v_i$$

$$\underline{\delta} = \max_i \underline{\delta}_i$$

Credible Equilibrium?

- Strategies used for the proof require all the opponents to play the minmax profile → can be costly → are the opponents threats credible
- The chosen strategies are not subgame perfect
- Do the conclusions apply to the payoffs of perfect eq.
  - YES. (perfect folk theorem)
- Nash –threats folk theorem: Let α^* be a static equilibrium (an equilibrium for one stage of the game) with payoffs *e*. Then for any $v \in V$ with $v_i > e_i$, for all players i, there is a $\underline{\delta}$, such that for all $\delta > \underline{\delta}$ there is a subgame perfect equilibrium of G(δ) with payoffs *v*.

Classic strategies repeated games

Strategy	Description	Advantages and Disadvantages
ALLC	Always cooperate	Susceptible to exploitation
ALLD	Always defect	No cooperation
Tit For Tat (TFT)	Cooperate on the first stage of the game, then do as the other player previously did	Highly robust as a general strategy but when playing against another TFT, cannot recover from an erroneous defection.
Contrite Tit For Tat (CTFT)	Both players start with "good standing." Cooperate if your opponent is in good standing, or if you are not. Otherwise defect.	Maintains a record of an opponent's "standing." Can recover from an opponent's erroneous defection
Generous Tit For Tat (GTFT)	As TFT but cooperate after an opponent's defection with a certain probability	Superior to TFT because it can recover from an erroneous defection. Exploitable by ALLD

Classic strategies repeated games

Strategy	Description	Advantages and Disadvantages
PAVLOV	Cooperate if and only if both protagonist and opponent played identically in the last round	Adapts by changing strategies when unsuccessful.
Prudent PAVLOV (P-PAVLOV)	Similar to PAVLOV, but only resume cooperation after two rounds of mutual defection	Can recover from an erroneous defection.
REMORSE	Cooperate if in "bad standing" or if both players cooperated in the last round	Maintains a record of an opponents "standing." Can recover from an opponent's erroneous defection
Suspicious Tit For Tat (STFT)	Defect on the first move, otherwise do as the other player last did	If plays against TFT the result is continual defection thereafter

Additional Strategies

Strategy	Descripton	Advantages and Disdvantages
Tit For Two Tats (TF2T or TFTT)	Cooperate on first move and defect after two consecutive defections by the opponent	Exploitable by a strategy which alternately cooperates and defects
GRIM	Cooperate if both players cooperated previously. Change to ALLD if the other player defects.	Unforgiving. Cannot recover from an erroneous defection

Games of Incomplete Information

- Recall:
  - Games of perfect information – sequential games
  - Games of imperfect information – simultaneous move games
  - Games of incomplete information?
    - Some players do not know the payoffs of the others

Bayesian game characterization

- Type of the players → contains any initial private information that a player might have
- Knowledge about the types is characterized by a pdf

$$p(\theta_1, \theta_2, \ldots, \theta_I)$$

$p(\theta_{-i} \mid \theta_i)$= player's 1 probability about its opponents types, given its own type

$$p(\theta_i) > 0, \quad \forall \theta_i \in \Theta_i$$

- Given the pure strategy space S_i, the payoff function of each player i, will depend on players' types:

$$u(s_1, \ldots, s_I, \theta_1, \ldots, \theta_I)$$

Bayesian Equilibrium

- **Definition:** A Bayesian equilibrium in a game of incomplete information with a finite number of types θ_i for each player i, prior distribution p, and pure strategy spaces Si is a Nash equilibrium of the "expanded game", in which each player's i space of pure strategies is the set $S_i^{\theta_i}$ of maps from θ_i to S_i.

Define a strategy profile: s(.) and $s'_i \in S_i^{\theta_i}$

The profile s(.) is a pure strategy Bayesian equilibrium if, for each player i

$$s_i(.) \in \arg\max_{s'_i(.) \in S_i} \sum_{\theta_i} \sum_{\theta_{-i}} p(\theta_i, \theta_{-i}) u_i(s'_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i, \theta_{-i})$$

$$s_i(.) \in \arg\max_{s'_i(.) \in S_i} \sum_{\theta_{-i}} p(\theta_{-i} \mid \theta_i) u_i(s'_i, s_{-i}(\theta_{-i}), \theta_i, \theta_{-i})$$

Example: Cournot competition

- Firms select the quantities of production si = qi
- Payoffs defined as

$$u_i = q_i(\theta_i - q_i - q_j)$$

- Common knowledge: firm 1 has type $\theta_1 = 1$
- Firm 2 – private information about θ_2
- Firm one beliefs:

$$\theta_2 = 3/4, \quad p = 1/2$$
$$\theta_2 = 5/4, \quad p = 1/2$$

- Belief of Firm 1 is common knowledge
- Firms choose their outputs simultaneously

Cournot competition: equilibrium

- q_1 = firm ones output
- For firm 2: for $\theta_2 = 3/4, \quad q_2^L$

  $\theta_2 = 5/4, \quad q_2^H$

$$\frac{\partial u_2}{\partial q_2} = 0 \Rightarrow q_2(\theta_2) = \frac{\theta_2 - q_1}{2}$$

$$\frac{\partial u_1}{\partial q_1} = 0 \Rightarrow \frac{1}{2} q_1 (1 - q_1 - q_2^H) + \frac{1}{2} q_1 (1 - q_1 - q_2^L) = 0 \Rightarrow q_1 = \frac{2 - q_2^H - q_2^L}{4}$$

Unique Bayesian equilibrium: $(q_1 = 1/3;\ q_2^L = 11/24;\ q_2^H = 5/24)$

Example

- An industry with 2 firms: incumbent (player 1) and potential entrant (player 2)
  - Player 1: Build new plant ?
  - Player 3: Enter?

	Enter	Don't
Build	0,-1	2,0
Don't	2,1	3,0

Building cost HIGH

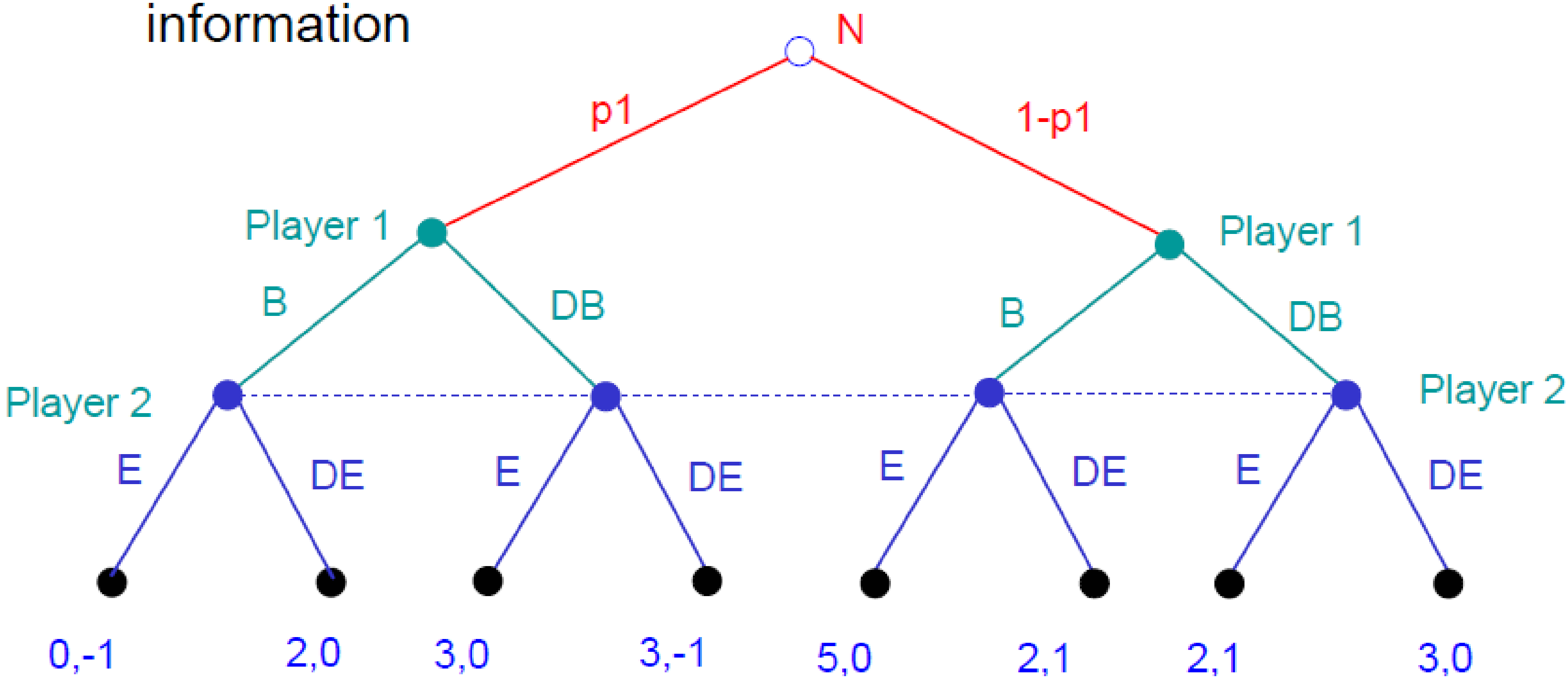
	Enter	Don't
Build	3,-1	5,0
Don't	2,1	3,0

Building cost LOW

Player 1 knows its cost for building (HIGH or LOW)
Player 2 does not know
→ assign a probability p1 for HIGH

Equivalent game

- Introduce prior move by nature: determines player 1 "type"
- Player's 2 incomplete information → imperfect information



Equilibrium solution

	Enter	Don't
Build	0,-1	2,0
Don't	2,1	3,0

Building cost HIGH

	Enter	Don't
Build	3,-1	5,0
Don't	2,1	3,0

Building cost LOW

If cost is HIGH: Don't build → dominant strategy for player 1

If cost is LOW; → dominant strategy for player 1: build

How about player 2?
- strategy for 2:

$$\text{enter if}: -1*(1-p_1)+1*p_1>0 \Rightarrow p_1>1/2$$

Equilibrium solution

	Enter	Don't
Build	0,-1	2,0
Don't	2,1	3,0

Building cost HIGH

	Enter	Don't
Build	1.5,-1	3.5,0
Don't	2,1	3,0

Building cost LOW

If cost is HIGH: Don't build → dominant strategy for player 1

If cost is LOW; no dominant strategy
→ y = probability that player 2 enters
→ x = probability that player 1 builds (given the type of player 1)

Building better than not building: $1.5y+3.5(1-y) > 2y+3(1-y) \rightarrow y< ½$
Enter better than not enter: $(-1)\, x(1-p_1) + 1[1-x(1-p_1)]>0$
$\rightarrow x<1/[2(1-p_1)]$