Game Theory

Department of Electronics

EL-766

Fall 2015

Dynamic games of complete information

- Extensive form games
  - In the examples studied so far, players choose their actions simultaneously
  - Can the strategic form game model the situation in which the order in which players move influences the outcome of the game?
  - Recall strategic (normal) form game characterized by three elements:
    - The set of players: {1, 2, …, I } (finite set)
    - The pure strategy space for each player i: S_i
    - Payoff (utility functions) for each profile of strategies: **s** = (s_1, …, s_I)

Example: Cournot vs. Stackelberg equilibrium

- Actions: choices of output levels: q1 and q2
- **Cournot:** both players choose their actions simultaneously, i.e., try to simultaneously maximize their utility functions
  - Example:

$$u_i(q_1,q_2)=[12-(q_1+q_2)]q_i \Rightarrow \begin{cases} \dfrac{\partial u_1(q_1,q_2)}{\partial q_1}=0 \Rightarrow r_1(q_2)=6-\dfrac{q_2}{2} \\ \dfrac{\partial u_2(q_1,q_2)}{\partial q_2}=0 \Rightarrow r_2(q_1)=6-\dfrac{q_1}{2} \end{cases}$$

- **Stackelberg:** player 1 chooses first, then player 2 observes the output q1, and consequently chooses q2
  - Is it the same equilibrium?
  - For which player this game is more advantageous?

Stackelberg equilibrium: cont.

- Player 2 sees q1, computes r2(q1) in the same fashion as before
- Player 1: knows that player 2 will maximize its utility based on q1, can compute r2(q1), and then maximize its utility by appropriately selecting q1.

$$u_i(q_1,q_2)=[12-(q_1+q_2)]q_i \Rightarrow \begin{cases} \dfrac{\partial u_2(q_1,q_2)}{\partial q_2}=0 \Rightarrow r_2(q_1)=6-\dfrac{q_1}{2} \\ u_1(q_1,q_2)=q_1\left(6-\dfrac{q_1}{2}\right) \\ \dfrac{\partial u_1(q_1)}{\partial q_1}=0 \Rightarrow q_1^*=6 \end{cases}$$

Stackelberg equilibrium: cont

- The resulting equilibrium point: $q^*_1=6$, $q^*_2=3$, with payoffs (18,9)
- Obtained by backward induction
- Cournot equilibrium: $q^C_1=4$, $q^C_2=4$, with payoffs (16,16)

- Leader has the advantage
- Other possible equilibria?
- Maybe, but not credible: would rely on empty threats from player 2, to maintain a different level q2

Extensive form games

- Stackelberg game (leader-follower) – example of a game in which players move sequentially and the order of the players' moves matters
- Multi-stage game
- Game of perfect information: exactly one player moves at a given stage, all the others have the one element choice: "do nothing"
- What is an extensive form game?

Extensive form games

The extensive form of a game contains the following inf:

- The set of players $i \in \mathbf{I}$
- The order of moves: Game tree
- The players' payoffs as a function of the previous moves
- What are the players choices when they move
- What each player knows when he makes its choice
- The probability distribution over any exogenous events

Exogenous events: moves by nature

Characterizing previous moves: definitions and notations

- Multi-stage game:
  - Players move simultaneously at stage k (do not know the actions of their opponents for stage k)
  - Know all the actions chosen at previous stages: 0,1,2,…, k-1.
  - Particular case: Stackelberg example (2 stage game) – at one stage just one player moves, the other one has action “do nothing”.
- At stage k, i-th player chooses an action from the choice set $A_i\left(\mathbf{h}^k\right)$

$\mathbf{h}^k = \left(\mathbf{a}^0, \mathbf{a}^1, \ldots, \mathbf{a}^k\right)$ = the history at the end of stage k

$\mathbf{a}^k = \left(a_1^k, a_2^k, \ldots, a_I^k\right)$ = stage k strategy profile

Game begins at stage 0, with $\mathbf{h}^0 = \varnothing$

More notations

- H^k = the set of all stage k histories
- Z = the set of terminal histories

- Player's i payoff represented as

$$u_i : H^K \to R$$

- For many applications: the payoff is additive over the game stages (weighted sum)

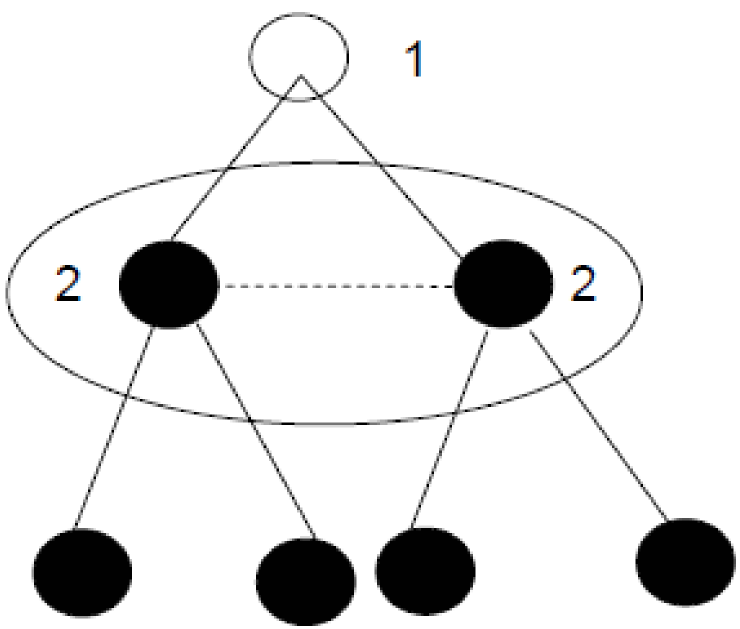
- Single stage payoffs for user i at stage k: $g_i\left(a^k\right)$

Game tree representation

- Components of a game tree
  - Vertices (nodes) – represent a particular history
    - Usually represented as ●
    - Exception: node with no history (first stage) ○
    - Labeled with the user id
  - Edges: correspond to the actions taken
    - Labeled with the action element
  - Simultaneous moves can also be modeled
  - When a node uncertain of past history (particular case is simultaneous moves, a dashed line unites its vertices for that stage. Sometimes this is represented also by encircling the vertices with an ellipse

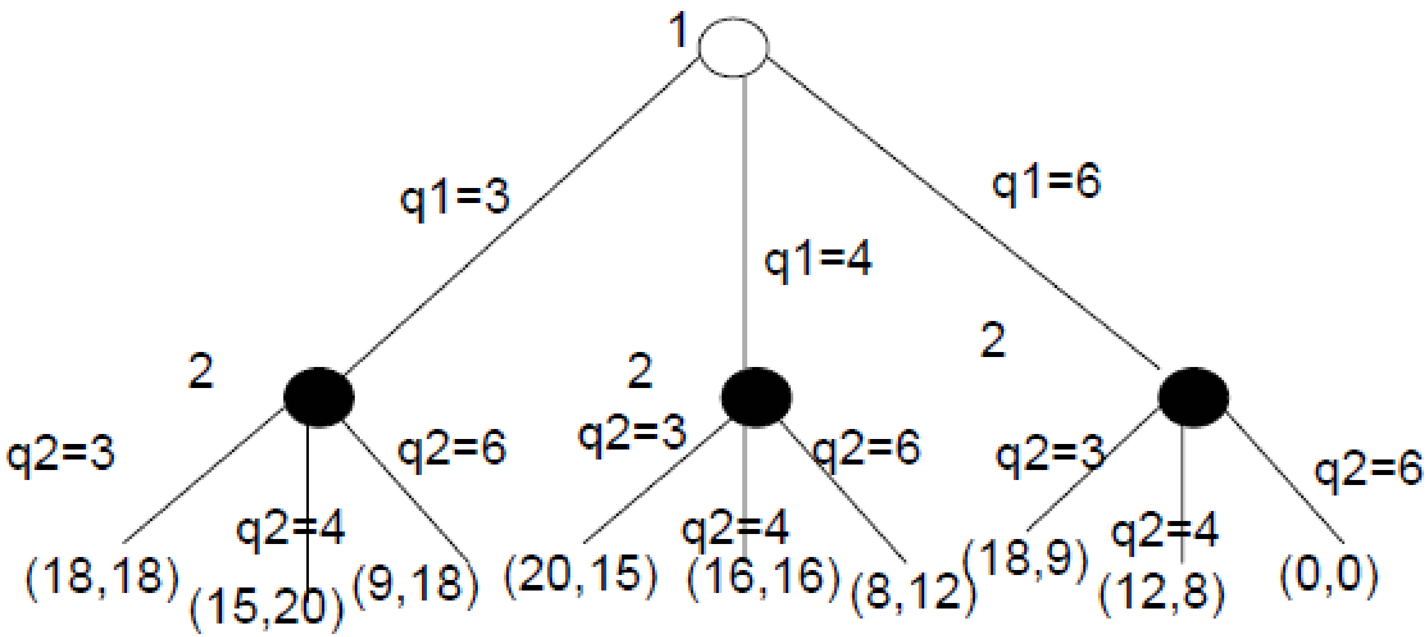
Game tree representation – cont.

- Time progresses in one direction: typically from left to right, or top to bottom.

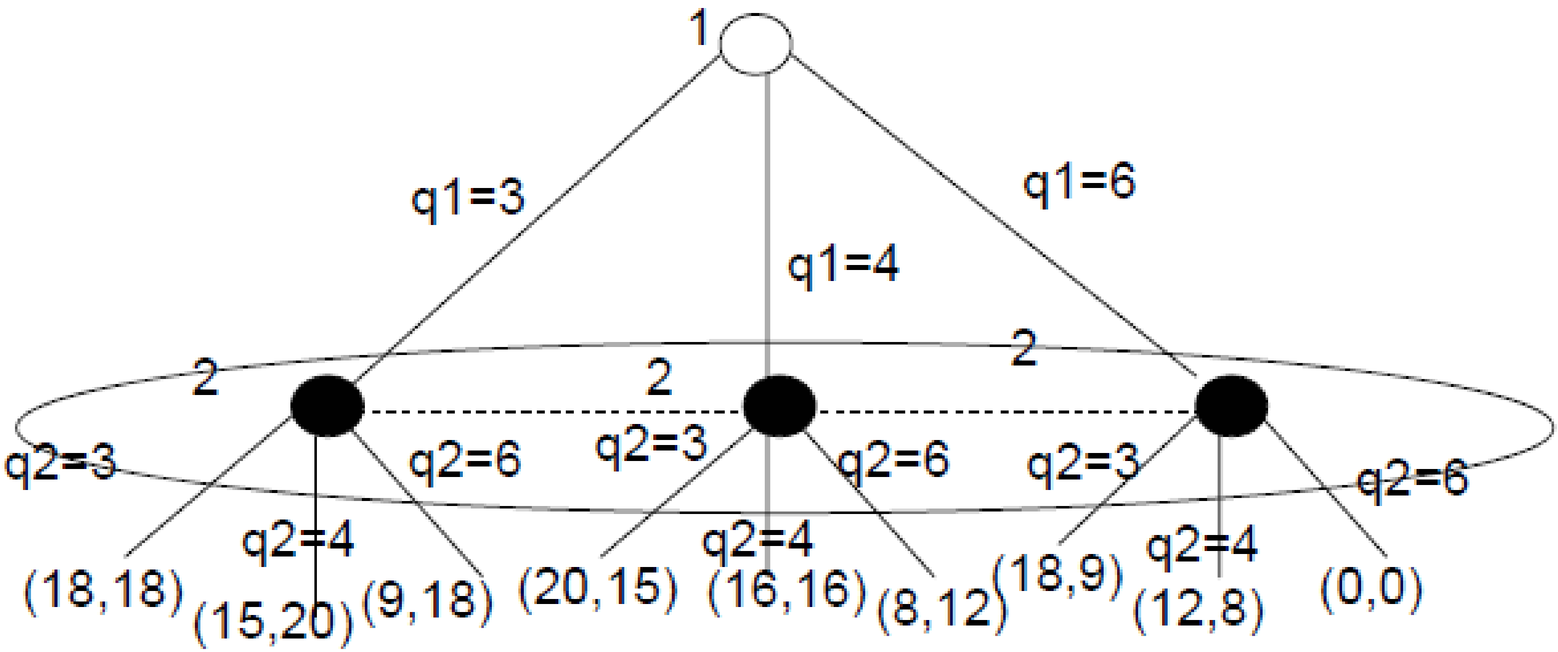


Stackelberg game representation

- Simplifying assumption: each player has only three possible output levels: 3, 4, 6



Cournot equivalent



- Is this representation unique?

Strategies and equilibria for extensive form games

- H_i = player i information sets
- $A_i = \bigcup_{h_i \in H_i} A(h_i)$ = set of all actions for player i
- A pure strategy is a map

$$s_i : H_i \to A_i, \quad s_i(h_i) \in A(h_i), \quad \forall h_i \in H_i$$

- Player i pure strategy space, is the space of all si
- The number of player's i pure strategy:

$$\# S_i = \prod_{h_i \in H_i} \# A(h_i)$$

- Path of s = information sets that are reached with positive probability
- Pure strategy Nash equilibrium: a strategy profile s*, such that each player's i strategy (s_i*) maximizes his expected payoff, given the strategies of his opponents (s_{-i}*)

Stackelberg game example

- How many pure strategies for players?