# Game Theory

Department of Electronics

EL-766

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# Which equilibrium is the best?

- **Pareto efficiency:**
  - A strategy profile is *Pareto optimal* if some players must be hurt in order to improve the payoff of other players
- Definition: A strategy profile *s** is said to be *Pareto optimal* iff there exists no other strategy profile *s',* such that if for some *j*

$$u_j(s') > u_j(s^*), u_i(s') \geq u_i(s^*), \forall i \in I \setminus J$$

- **Observations:**
  - A strategy profile that is a Nash equilibrium may not be Pareto optimal (efficient).
  - A strategy profile which is Pareto efficient, is not necessarily a Nash equilibrium.
  - We would like Nash equilibrium to be Pareto efficient.

# Pareto Efficient: Example Game

| | $a_1$ | $a_2$ |
|---|---|---|
| $a_1$ | 2,3 | -2,7 |
| $A_2$ | 6,-5 | **0,-1** |

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Pareto Efficient: _____

# Cournot Example

- Duopoly producing a product
- Strategies: quantities produced

$$q_i \in Q_i = [0, \infty]$$

- Price of the goods:

- $$u_i(q_1, q_2) = q_1 p(q) + c_i(q_i)$$

- Utilities:

$$p(q), where \Rightarrow q = q_1 + q_2$$

- Cournot reaction functions: specify each firm's optimal output for each fixed output level of its opponent

$$r_1 : Q_2 \rightarrow Q_1$$

- Max. utility:

$$r_2 : Q_1 \rightarrow Q_2$$

$$p(q_1 + r_2(q_1)) + p^{'}(q_1 + r_2(q_1))r_2(q_1) - c_2^{'}(r_2(q_1)) = 0$$

# Cournot Example continued

- For
$$p(q) = \max(0, 1 - q)$$
$$c_i(q_i) = cq_i, 0 \le c \le 1$$

- The reaction functions become
$$r_2(q_1) = (1 - q_1 - c)/2$$
$$r_1(q_2) = (1 - q_2 - c)/2$$

- And the Nash equilibrium is:
$$\left. \begin{array}{l} q_1^* = r_2(q_1^*) \\ q_2^* = r_1(q_2^*) \end{array} \right] \Rightarrow q_1^* = q_2^* = (1 - c)/3$$

# How to get to Nash equilibrium?

- Up till now: Introspection (observation or examination ) and deduction
- Learning or evolution
- Cournot example: players take turns setting their outputs and each player's output is a *best response* to the output his opponent chose in the previous period.

$$q_1^0, q_2^1 = r_2(q_1^0), q_2^1 = r_1(q_2^1) = r_1(r_2(q_1^0)), ....$$

- If it reaches steady state, then

$$\left.\begin{array}{l} q_1^* = r_2(q_1^*) \\ q_2^* = r_1(q_2^*) \end{array}\right] \Rightarrow q_1^* = q_2^* = (1-c)/3$$

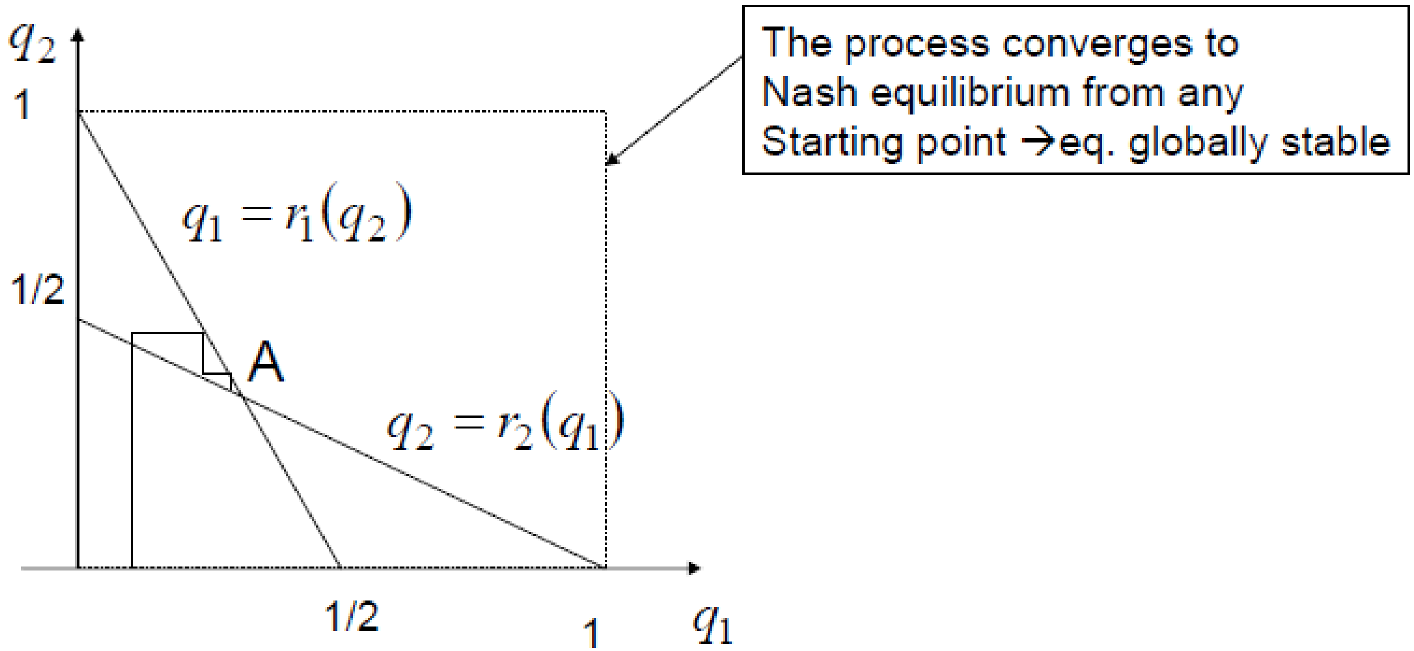
# Asymptotically stable equilibrium

- Learning-type adjustment process need not converge to a steady state
- If a process converges to a particular steady state for all initial points sufficiently close to it, asymptotically stable
- Cournot example, asymptotically stable:

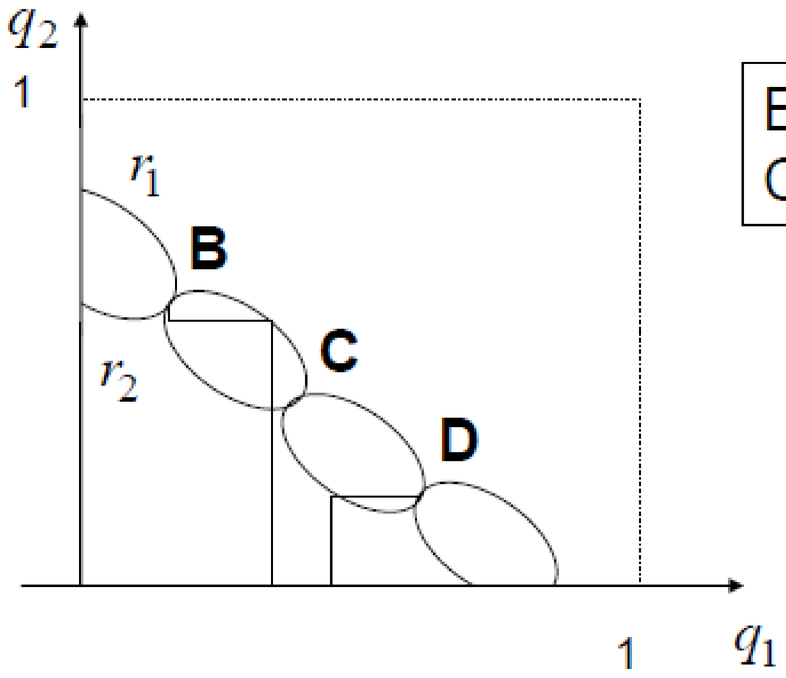
$$\left.\begin{array}{l} p(q) = 1 - q \\ c_i(q_i) = 0 \\ Q_i = [0,1] \end{array}\right] \Rightarrow \left\{\begin{array}{l} r_i(q_j) = (1 - q_j)/2 \\ \text{Nash eq. } A = \left(\frac{1}{3}, \frac{1}{3}\right) \end{array}\right.$$

# Cournot example: evolution

- Unique Nash equilibrium is at the intersection of the reaction curves



# Asymptotic stability: continued




B, D = stable Nash eq.
C = Nash eq, but not stable

Stability of Nash equilibrium depends on the slope of the reaction functions

# Asymptotic stability condition

- Slope of the reaction function:

$$\frac{dr_i}{dq_j} = -\frac{\partial^2 u_i}{\partial q_i \partial q_j} / \frac{\partial^2 u_i}{\partial q_i^2}$$

- Sufficient condition for equilibrium (in an open neighborhood of the Nash equilibrium):

$$\left|\frac{dr_1}{dq_2}\right|\left|\frac{dr_2}{dq_1}\right| < 1 \Leftrightarrow \frac{\partial^2 u_1}{\partial q_1 \partial q_2}\frac{\partial^2 u_2}{\partial q_1 \partial q_2} < \frac{\partial^2 u_1}{\partial q_1^2}\frac{\partial^2 u_2}{\partial q_2^2}$$

- Note: Same stability condition if the firms react simultaneously instead of alternatively to their opponent's current actions.

# Example: Shapley cycle

- Best response adjustment process does not necessarily converge

| | L | M | R |
|---|---|---|---|
| U | 0,0 | 4,5 | 5,4 |
| M | 5,4 | 0,0 | 4,5 |
| D | 4,5 | 5,4 | 0,0 |

# Adjustment process

- A form of repeated game play
- Players ignore the effect of their current actions on the opponent's future actions
- Myopic play = a repeated game in which there is no communication between players, no memory of past events, or prediction of future payoffs.
- The adaptation is based on the current state of the game.
- Two convergence dynamics possible in a myopic game
  - Best response dynamic
  - Better response dynamic
  - Both require additional conditions to ensure convergence.

# Existence of Nash equilibrium

- Theorem: Every finite strategic-form game has a mixed strategy equilibrium.
- Before giving a proof, we need some functional analysis basics
- Convex set: A set *S* in n-dimensional space is called a convex set, if the line segment joining any pair of points of S lies entirely in S.

# Some more functional analysis definitions

- Compact set: A bounded set $S$ is compact if there is no point $x \notin S$ such that the limit of a sequence formed entirely from elements in $S$ is $x$.
- –Compact set, closed and bounded
- –Examples: any closed finite interval [a, b], closed disc, etc.
- –Not compact: (a, b], [a, ∞)

# Some more functional analysis definitions

- Continuous function
- A function $f$: $X \to Y$ is continuous if for all $x_0 \in X$ the following conditions hold:

$$f(x_0) \in Y$$

$$\lim_{x \to x_0} f(x) \in Y$$

$$\lim_{x \to x_0} f(x) = f(x_0)$$

# Nash equilibrium for infinite games with continuous payoffs

- Debreu's theorem:Consider a strategic-form game whose strategy spaces $S_i$ are non-empty, compact, convex subsets of an Euclidian space. If the payoff functions $u_i$ are continuous in s, and quasi concave in $s_i$, there exists a pure strategy Nash equilibrium.

- Glicksberg theorem: Consider a strategic-form game whose strategy spaces $S_i$ are nonempty compact subsets of a metric space. If the payoff functions $u_i$ are continuous then there exists a Nash equilibrium in mixed strategies.