Nash Equilibrium

- Many problems (especially in resource allocation) are not solvable by iterated strict dominance.
- A broad class of games are characterized by the Nash equilibrium solution.
- Nash equilibrium is a profile of strategies such that each player strategy is an optimal response to the other players' strategies.
- Definition: A mixed strategy profile $\sigma^*$ is a Nash equilibrium, if for all players $i$

$$u_i = (\sigma_i^*, \sigma_{-i}^*) \geq (s_i, \sigma_{-i}^*), \forall s_i \in S$$

- Similar def. for the pure strategies

- Note: If a player uses a nondegenerate strategy in a Nash eq., he must be indifferent between all the pure strategies to which he assigns positive probability.
Strict Nash equilibrium

- NE is strict if each player has a unique best response to his rivals’ strategies
  - No mixed strategies
- \( s^* \) is a strict Nash equilibrium iff is a Nash equilibrium, and for all \( i \), and all \( s_i \)

\[
s_i \neq s_i^* \implies u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*)
\]

- Existence
  - Pure strategy Nash equilibrium may not exist
- Uniqueness
  - Nash equilibrium need not to be unique
- Efficiency
  - Pareto Optimality?
Example Games

- Prisoners Dilemma
- Auctions
- Traveling Salesman
- Matching Pennies
- Friend or Foe
Example Games: Matching pennies

- 2 players
  - simultaneously announce head or tails.
- If match
  - player 1 wins, otherwise player 2 wins.
  - Equilibrium: randomize with prob 1/2

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Game: Friend or Foe

- **Game**: Game show: prize 1000$.
- To win the money, the 2 members of the team must go in separate rooms and vote Friend or Foe.
- It is assumed that the two team mates are complete strangers.
- If both choose Friend, then the two players split the money equally.
- If one chooses Foe, while the other chooses Friend, then the former takes all the money.
- If both choose Foe, then both players lose, and they get nothing.
Friend or Foe: Nash equilibrium

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<td>Foe</td>
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How to get to the Nash Equilibrium?

• Rational Introspection
• Focal Point
• Trial and Error
• Pre-play communication
Which equilibrium is the best?

- **Pareto efficiency:**
  - A strategy profile is *Pareto optimal* if some players must be hurt in order to improve the payoff of other players.
- **Definition:** A strategy profile $s^*$ is said to be *Pareto optimal* if for there exists no other strategy profile $s'$, such that if for some $j$

  $$ u_j(s') > u_j(s^*), u_i(s') \geq u_i(s^*), \forall i \in I \setminus J $$

- **Observations:**
  - A strategy profile that is a Nash equilibrium may not be Pareto optimal (efficient).
  - A strategy profile which is Pareto efficient, is not necessarily a Nash equilibrium.
  - We would like Nash equilibrium to be Pareto efficient.
Pareto Efficient: Example Game

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Pareto Efficient: _____